Pricing kernels and their dependence on the implied volatility index

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Statement of Authorship

I hereby confirm that I have authored this Master's thesis independently and without use of others than the indicated sources. All passages which are literally or in general matter taken out of publications or other sources are marked as such.

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Abstract

Pricing kernels are crucial for understanding investor attitude toward market risk. According to classical economic theory, pricing kernel must be positive and decreasing as a function of aggregate resources. Nevertheless, several empirical studies revealed nonmonotonicity of the empirical pricing kernels and this phenomenon is refereed to as the empirical pricing kernel puzzle. In this thesis nonparametric estimates of the pricing kernels conditional on the implied volatility index are proposed. Obtained results show that conditioning on a suited level of volatility leads to decreasing pricing kernels. Also skewness of risk neutral and physical densities was related to the shape of the pricing kernels.

Keywords: Empirical pricing kernel, implied volatility, risk neutral density, physical density, local constant kernel regression, local linear kernel regression, risk aversion

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1 Introduction

The understanding of investor attitude toward market risk is one of the crucial tasks in modern quantitative finance. A notion of the empirical pricing kernel is directly involved in the investigation of such a problem, since there is an explicit connection between pricing kernel and investor utility function that defines investor risk patterns, see Kahneman and Tversky (1979), Leland (1980) and Jackwerth (2000).

Under assumptions of classical economic theory (e.g., Cochrane (2009)), the pricing kernel should be positive and decreasing as a function of aggregate resources, which implies the risk-averse behavior of investors. Nevertheless, empirical studies reveal that pricing kernels can be locally nondecreasing, see Jackwerth and Rubinstein (1996), Chabi-Yo (2008) and Christoffersen et al. (2013). This phenomenon is often referred to as the pricing kernel puzzle, and investigation of the possible reasons leading to such an phenomenon is the main aim of this research.

Estimation of pricing kernels from option prices and historical returns of the underlying security was firstly introduced in Aït-Sahalia and Lo (2000), Jackwerth (2000) and Rosenberg and Engle (2002). The approach proposed by Aït-Sahalia and Lo (2000) for estimation of the pricing kernels relies on the consideration of pricing kernel as a ratio of risk neutral density obtained from the derivative market and physical density estimated using the historical time series of underlying asset. Empirical pricing kernels obtained by Aït-Sahalia and Lo (2000), Jackwerth (2000) and Rosenberg and Engle (2002) have nonmonotonic shape. Moreover, obtained estimates of pricing kernels are hump-shaped or U-shaped depending on time span and data sets. Nevertheless, no uniform bands have been provided for obtained pricing kernel estimates, so it can not be decided whether nonmonotonicty comes due to statistical noise or is a real fact. The estimation procedure of Adesi et al. (2014) has the same disadvantage. To tackle such a problem, Härdle et al. (2015) developed a uniform confidence band of the empirical pricing kernel. Moreover, Golubev et al. (2014) proposed a test to verify monotonicity of the empirical pricing kernel. As a consequence, the hypothesis of monotone decreasing pricing kernel has been rejected at 5% and at 10% significance level for German DAX data in years 2002 and 2000 respectively. Another formal statistical test of pricing kernel monotonicity has been proposed by Beare and Schmidt (2016), and applied to option data for the $S\&P$ 500 index. As a result, the null hypothesis of pricing kernel monotonicity was rejected. Thus, observed nonmonotonicity of pricing kernels can hardly be related to statistical noise, and is an established feature of the empirical pricing kernel.

A large body of literature investigates reasons leading to empirical pricing kernels puzzle. Microeconomic views explaining the puzzle are presented in Grith et al. (2016). Hens and Reichlin (2012) investigate how violations of assumptions of standard expected utility models can influence the form of the pricing kernel. A comprehensive overview of the literature investigating reasons for the pricing kernel puzzle are provided in Grith et al. (2016).

Market volatility is one of the most important factors influencing the pricing of assets. Therefore, volatility has been considered as a state variable for estimation of the pricing kernel in several recent researches, see Song and Xiu (2016) and Chabi-Yo (2012).

In this study nonparametric estimates of the pricing kernels conditional on the implied volatility index are proposed. Specifically, volatility indexes provided by Deutsche Börse AG have been used, and, contrary to Song and Xiu (2016), term structure of volatility indexes was taken into account. European options on the DAX 30 index have been used to conduct the empirical study. Obtained results show that conditioning on a suited level of implied volatility provides a decreasing pricing kernel. Moreover, relationship between market volatility and U-shaped and hump-shaped forms of the pricing kernel was investigated.

The paper is structured as follows. In Section 2, we discuss a strategy for estimation of pricing kernels. Nonparametric estimation of risk neutral density is discussed in Section 3, whereas nonparametric methods for estimation of physical density are presented in Section 4. Empirical estimates of risk neutral and physical densities as well as pricing kernels are provided in Section 5. Moreover, the relationship between the implied volatility index and shape of pricing kernels is analyzed in Section 5. In Section 6, we conclude and summarize obtained results.

2 Empirical pricing kernel estimation

Let us consider a risky security with the price process $\{S_t, t \in [0,T]\}$. The risk-free interest rate process $\{r_t, t \in [0,T]\}$ is assumed to be deterministic. It is further assumed that the market is complete. Then due to the Second Fundamental Theorem of Asset Pricing a unique martingale measure Q exist, which can be used for valuation of derivatives. In particular, let us denote by P_t the price at time t of any contingent claim with maturity T and payoff $\psi(S_T)$, then

$$P_t = \mathrm{E}^{\mathrm{Q}}\left[e^{-r_{t,\tau}}\psi(S_T)\right], \tag{1}$$

where τ is time to maturity and E^{Q} denotes the expectation under risk neutral measure Q.

By changing the risk neutral measure to physical measure, time-t price of derivative with payoff $\psi(S_T)$ can be rewritten in such a form:

$$P_t = \mathrm{E}^{\mathrm{Q}}\left[e^{-r_{t,\tau}}\psi(S_T)\right] = \mathrm{E}\left[e^{-r_{t,\tau}}\psi(S_T)\,\mathcal{K}(S_T)\right], \tag{2}$$

where $\mathcal{K}(S_T)$ denotes the pricing kernel at time t, E denotes the expectation under the physical measure (also known as historical measure) P.

Having written (2) more explicitly, one obtains such an equation:

$$P_t = \int_0^{\infty} e^{-r_{t,\tau}}\psi(x)\,q(x)\,dx = \int_0^{\infty} e^{-r_{t,\tau}}\psi(x)\,\mathcal{K}(x)\,p(x)\,dx, \tag{3}$$

where p and q are the probability density functions of the physical and risk neutral measures respectively.

It can be seen from (3) that

$$\mathcal{K} = \frac{q}{p} \tag{4}$$

Representation of the pricing kernel (4) sheds light on one way of pricing kernel estimation, which was implemented in this paper. Therefore, pricing kernel has been estimated by a ratio of estimates of risk neutral density and physical density:

$$\widehat{\mathcal{K}} = \frac{\widehat{q}}{\widehat{p}} \tag{5}$$

2.1 Model setup

Let us start by introducing notation that is used in this work. The current date is denoted by t, and the maturity date of the option is denoted by T. As a consequence, $\tau = T - t$ denotes time to maturity. Value of the DAX 30 index at time t is denoted by S_t. Furthermore, volatility at time t for options with time to maturity τ is denoted by V_t^τ. State price density and physical density are denoted by q_{S_T} and p_{S_T} respectively. Although, for convenience, one drops the index and just uses q and p.

VDAX-NEW index has been used as a proxy for volatility of S_t. Especially, we used subindixes of VDAX-NEW depicting different option maturities. Volatility subindixes are denoted by Z_t^τ, where superscript τ reflects the time to maturity. Due to the fact, that V_t^τ can not be directly observed from the market, it is replaced with Z_t^τ.

For the purpose of investigation of the dependence between pricing kernels and implied volatility index, it is reasonable to estimate pricing kernels conditional on volatility index and time to maturity $\widehat{\mathcal{K}}\left(r|\tau, Z_t^\tau\right)$. Such an estimate can be obtained from (4)

$$\widehat{\mathcal{K}}\left(r|\tau, Z_t^\tau\right) = \frac{\widehat{q}\left(r|\tau, Z_t^\tau\right)}{\widehat{p}\left(r|\tau, Z_t^\tau\right)} \tag{6}$$

Thus, estimation of conditional risk neutral and physical density is required to obtain estimate of conditional pricing kernel. Methodology for estimation of risk neutral and physical densities is presented in Section 3 and Section 4 respectively.

3 State price density estimation

3.1 Estimation strategy

Price of European call option (see, e.g., Aït-Sahalia and Lo (1998)) can be expressed as

$$C\left(S_t, K, \tau, r_{t,\tau}, V_t\right) = e^{-r_{t,\tau}} \int_0^{\infty} \max\left(S_T - K, 0\right) q\left(S_T | S_t, K, \tau, r_{t,\tau}, V_t\right) dS_T, \tag{7}$$

where

- S_t is the price of the underlying security at time t,
- K is the exercise price of an option,
- T is the expiration date of an option,
- τ is the time to maturity $\tau = T - t$,
- $r_{t,\tau}$ is the risk free interest rate at time t for time to maturity τ,
- V_t is the market volatility.

Due to result of Breeden and Litzenberger (1978), state price density can be recovered from the second order derivative of option price with respect to strike.

$$q\left(S_T | S_t, K, \tau, r_{t,\tau}, V_t\right) = e^{r_{t,\tau}} \left. \frac{\partial^2 C\left(S_t, K, \tau, r_{t,\tau}, V_t\right)}{\partial K^2} \right|_{K=S_T} \tag{8}$$

Nevertheless, estimation of state price density can not be done in practice since V_t is not observable. To tackle such a problem, state variable V_t is replaced by the observable Z_t. Thus, formula (8) for calculation of the state price density can be rewritten in such a way

$$q\left(S_T | S_t, K, \tau, r_{t,\tau}, Z_t\right) = e^{r_{t,\tau}} \left. \frac{\partial^2 C\left(S_t, K, \tau, r_{t,\tau}, Z_t\right)}{\partial K^2} \right|_{K=S_T} \tag{9}$$

The major burden of nonparametric multivariate regression technique is the so called curse of dimensionality. Which essentially means that the rate of convergence of nonparametric estimate decreases rapidly as the number of regressors increases. The obvious solution to such a problem can be found in making some assumptions to be able to reduce the complexity of regression.

Following existing studies, for instance Aït-Sahalia and Lo (2000), let us assume that the call option pricing function is homogeneous of degree in S_t and K. In other words, following equality holds

$$C\left(K, \tau, Z_t, S_t\right) = S_t C\left(\frac{K}{S_t}, \tau, Z_t, 1\right). \tag{10}$$

Therefore, a new function depending only on four state variables can be defined:

$$\overline{C}\left(m, \tau, Z_t\right) = \frac{1}{S_t} C\left(K, \tau, Z_t, S_t\right), \tag{11}$$

where $m = \frac{K}{S_t}$ is the moneyness of an option.

Thus, calculation of the state price density can be done in such a way:

$$q\left(S_T | K, \tau, V_t\right) = \frac{1}{S_t} e^{r_{t,\tau}} \frac{\partial^2 \overline{C}}{\partial m^2}\bigg|_{m=\frac{S_T}{S_t}} \tag{12}$$

The density estimates are defined on the scale of S_T. To define the density on the scale of $R_T = \log\left(S_T/S_t\right)$ such a transformation must be applied $q\left(R_T\right) = q\left(S_T\right) S_T$. It should be mentioned that all results in this work are shown on the scale of $\log\left(S_T/S_t\right)$ returns.

3.2 Local linear regression

Construction of nonparametric estimators of option pricing function as well as its partial derivative with respect to moneyness will be presented in this section. We will consider a local linear regression of scaled option prices $\overline{C}$ on time to maturity τ, volatility z and moneyness m. It should be mentioned that one deals with cross-sectional regression in this case, because option prices with different strikes and maturities are being considered simultaneously. For any fixed $u = \left(\tau, z, m\right)^\top$ at which risk neutral density has to be estimated, such minimization problem is considered:

$$\min_{\alpha,\beta} \sum_{i=1}^{n} \left\{\overline{C}_i - \alpha - \beta^\top \left(u_i - u\right)\right\}^2 K_h\left(u_i - u\right), \tag{13}$$

where $u_i = \left(\tau_i, z_i, m_i\right)^\top$ is the characteristic of the $i-th$ option, $\overline{C}_i$ is the scaled option price.

Function K_h is a multivariate kernel function with a bandwidth $h = \left(h_\tau, h_z, h_m\right)^\top$:

$$K_h\left(u_i - u\right) = \frac{1}{h_\tau} K\left(\frac{\tau_i - \tau}{h_\tau}\right) \frac{1}{h_z} K\left(\frac{z_i - z}{h_z}\right) \frac{1}{h_m} K\left(\frac{m_i - m}{h_m}\right), \tag{14}$$

where univariate kernel K is chosen as the density of the standard normal distribution.

It worth mentioning that α is a local estimate of $\overline{C_i}$, whereas β provides local estimates of the first order partial derivatives $\frac{\partial \overline{C}}{\partial u}$. This fact can be easily established using Taylor expansion of function $\overline{C}$ and (13).

Furthermore, the solution of minimization problem (13) can be written using closed-form representation:

$$\begin{bmatrix} \widehat{\alpha} \\ \widehat{\beta} \end{bmatrix} = \left(\Omega^{\top} K \Omega\right)^{-1} \Omega^{\top} K \overline{C}, \tag{15}$$

where

$$\Omega = \begin{bmatrix} 1 & (u_1 - u)^{\top} \\ \vdots & \vdots \\ 1 & (u_n - u)^{\top} \end{bmatrix},$$

$$\overline{C} = \begin{bmatrix} \overline{C}_1 \\ \vdots \\ \overline{C}_n \end{bmatrix},$$

$$K = \begin{bmatrix} K_h(u_1 - u) & & \\ & \ddots & \\ & & K_h(u_n - u) \end{bmatrix}.$$

Therefore, nonparametric local linear estimator for call pricing function $\overline{C}(u)$ can be written as

$$\widehat{\overline{C}}(\tau, z, m) = \widehat{\alpha} = e_1^{\top} \left(\Omega^{\top} K \Omega\right)^{-1} \Omega^{\top} K \overline{C}, \tag{16}$$

where $e_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^{\top}$.

As mentioned before, the estimates of the first-order partial derivatives of $\overline{C}(\tau, z, m)$ are provided by vector β. Thus, the first-order derivative of option pricing function with respect to moneyness can be estimated by the third component of vector $\widehat{\beta}$

$$\widehat{\beta}_3 = e_4^{\top} \left(\Omega^{\top} K \Omega\right)^{-1} \Omega^{\top} K \overline{C}, \tag{17}$$

where $e_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^{\top}$.

The second-order partial derivative of option pricing function with respect to moneyness is required for estimation of risk-neutral density. That is why one needs to calculate $\frac{\partial \beta_3}{\partial m}$, and one of numeric algorithms can be used to achieve such a goal. In particular, finite difference approximation has been used in this study to compute the second-order derivative numerically. According to Hildebrand (1987) $\frac{\partial \beta_3}{\partial m}$ can be calculated in the following way:

$$\frac{\partial \beta_3}{\partial m} \approx \frac{\beta_3(m+h) - \beta_3(m-h)}{2h}, \tag{18}$$

where h represents a small change in variable of interest (moneyness in our case). Moreover, Hildebrand (1987) provides corresponding estimation error that is of order $\mathcal{O}\left(h^2\right)$. In this research $h = 0.001$ was used for numerical calculation of $\frac{\partial \beta_3}{\partial m}$. Thus, utilization of numerical procedure for calculation of $\frac{\partial \beta_3}{\partial m}$ leads to estimation error of order $\mathcal{O}\left(10^{-6}\right)$.

3.3 Bandwidth selection

Based on general theory (e.g., Härdle (2004))the optimal bandwidth for estimation of the call pricing function is of order $n^{(-1/(4+d))}$. Such bandwidth selection assures the optimal rate of convergence with regard to mean-squared criterion. Because of that, in practice, bandwidth can be chosen as $h_j^* = c_j \sigma_j n^{(-1/(4+d))}$, where σ_j is unconditional standard deviation of a regressor j. However, h_j^* cannot be computed in practice since constant c_j is unknown. The standard approach for identification of this constant is leave-one-out cross validation method (e.g., Härdle (2004) and Li and Racine (2004)), which selects bandwidth as a solution to such optimization problem

$$\min_h \frac{1}{n} \sum_{i=1}^{n} \left\{ \overline{C}_i - \widehat{\overline{C}}_{h,-i} \left(\tau_i, z_i, m_i\right) \right\}^2 \omega \left(\tau_i, z_i, m_i\right), \tag{19}$$

where $-i$ means leaving the ith observation out, ω is the weighting function. In this research we used implementation of leave-one-out cross validation approach that has been done according to Hayfield and Racine (2008) and Hastie et al. (2001).

3.4 Asymptotic theory

Let us denote the number of options in the sample by n. Then asymptotic distribution for a risk neutral density $q\left(r \mid \tau, z, m\right)$ estimated by local linear regression can be found, for instance, in Song and Xiu (2016) and has such a representation:

$$\begin{aligned} & n^{1/2} h_m^2 (h_\tau h_z h_m)^{1/2} \left(\widehat{q}\left(r \mid \tau, z, m\right) - q\left(r \mid \tau, z, m\right)\right) \\ & \xrightarrow{\mathcal{L}} \mathrm{N}\Bigg(0, m^2 \left[\int K^2(c) dc\right]^3 \times \left[\int \left(c\dot{K}^2(c) + K(c)\right)^2 dc\right] \\ & \times \frac{s^2\left(\tau, z, m\right)}{\left(\int K(c) c^2 dc\right)^2 \pi\left(\tau, z, m\right)}\Bigg), \end{aligned} \tag{20}$$

where

- $s^2\left(\tau, z, m\right)$ is the conditional variance for the local linear regression of C on the explanatory variables τ, z, and m,

- $\pi(\tau, z, m)$ is the joint density of the explanatory variables,
- h_τ, h_z and h_m are smoothing parameters,
- K is a univariate kernel function.

Thus, accurate estimation of conditional variance is crucial for calculation of asymptotic distribution of the risk neutral density. Method of Fan and Yao (1998) for estimation of the conditional variance has been adopted in this research, and this approach is briefly described below.

Let $\mu(u) = \mathrm{E}(C \mid u)$ and $\sigma^2(u) = \mathrm{Var}(C \mid u)$ be a regression function and a conditional variance respectively with $\sigma^2(u) > 0$. Firstly, $\mu(u)$ is estimated by the local linear technique, i.e., $\widehat{\mu}(u) = \widehat{a}$ if

$$(\widehat{a}, \widehat{b}) = \arg\min_{a,b} \sum_{i=1}^{n} \left\{C_i - a - b^T(u_i - u)\right\}^2 K_h(u_i - u), \tag{21}$$

where K is a kernel function, h is a bandwidth. Afterwards, $\sigma^2(u)$ can be estimated using nonparametric regression of squared fitting errors $(C_i - \widehat{m}(u_i))^2$ on state variables:

$$(\widehat{\alpha_1}, \widehat{\beta}_1) = \arg\min_{\alpha_1,\beta_1} \sum_{i=1}^{n} \left\{\widehat{r_i} - \alpha_1 - \beta_1^T(u_i - u)\right\}^2 W_h(u_i - u), \tag{22}$$

where $\widehat{r_i} = (C_i - \widehat{m}(u_i))^2$, W is a kernel function, h is a bandwidth.

As a consequence, estimate for conditional variance is given by

$$\widehat{\sigma}^2(u) = \widehat{\alpha}_1 \tag{23}$$

Therefore, asymptotic distribution of $\widehat{q}$ can be calculated using formula (20), which enables computation of confidence intervals for risk neutral density. It should be mentioned that the error from numerical differentiation (18) was not included in calculation of confidence intervals, since it is of order $\mathcal{O}\left(10^{-6}\right)$ in our setup and can be neglected.

4 Physical density estimation

Accurate estimation of the physical density is crucial for adequate estimation of the pricing kernel. Some approaches for recovering of the physical density from historical time series adopted by Aït-Sahalia et al. (2009) and Song and Xiu (2016) are presented in this section.

The goal is to estimate density of returns conditional on fixed time to maturity and volatility. In other words, $p(r|\tau, z)$ has to be estimated based on the past observations of returns r for a fixed time to maturity τ and for a fixed level of implied volatility index z. The first step in estimation procedure is to collect time series of returns and corresponding values of implied volatility:

$$\left(\log\left(\frac{S_{t-i}}{S_{t-i-\tau}}\right), z_{t-i}\right), \tag{24}$$

where $i = 1, 2, \ldots, k$.

Let us denote $\log\left(\frac{S_{t-i}}{S_{t-i-\tau}}\right)$ by $r_{t_i,\tau}$. Then estimation of $p(r|\tau, z)$ can be regarded as an estimation of $p(r_\tau|z)$, since time to maturity τ is taken into account by construction of r_τ.

Estimation of the conditional density can be connected with the nonparametric regression problem. Due to observation of Fan et al. (1996) the following hold as $h \to 0$

$$\mathrm{E}\left(K_h\left(R_\tau - r_\tau\right) \mid Z = z\right) \to p\left(r_\tau \mid z\right), \tag{25}$$

where K is a kernel function and $K_h(u) = \frac{1}{h}K\left(\frac{u}{h}\right)$. Thus, the term on the left-hand side of equation (25) can be considered as the regression of the variable $K_h(R_\tau - r_\tau)$ on Z. Having made a direct link between two concepts: conditional density and nonparametric regression, one can use different nonparametric regression techniques for the purpose of conditional density estimation. Specifically, local linear and local constant regressions have been used in this research for estimation of the conditional physical density.

Furthermore, it should be mentioned, that two approaches with regard to underlying data can be used for estimation of the conditional physical density. Let us imagine that we want to estimate physical density for year 2012. On the one hand, one can use only observations from this year (i.e., from 01.01.2012 till 31.12.2012) to estimate physical density. This procedure will be referred to as contemporaneous approach for estimation of the conditional physical density. On the other hand, one can use observations from the time interval before 1.01.2012 for estimation of physical density (e.g., from 01.01.2011 till 31.12.2011). This procedure will be referred to as historical approach for estimation of the conditional physical density. It is worth mentioning that only contemporaneous approach for estimation of the conditional physical density was utilized in this paper.

4.1 Local linear regression for conditional density estimation

Application of local linear regression for conditional density estimation, see Fan and Gijbels (1996), relies on the minimization of

$$\min_{\alpha,\beta} \sum_{i=1}^{k} \left\{ K_{h_r}\left(r_i - r\right) - \alpha - \beta\left(z_i - z\right) \right\}^2 K_{h_z}\left(z_i - z\right) \tag{26}$$

with regard to local parameters α and β. According to (25), α can be considered as the estimate of the conditional density.

$$\widehat{p}\left(r \mid \tau, z\right) = \widehat{\alpha} \tag{27}$$

Nevertheless, bandwidths h_r and h_z are not known. Moreover, inadequate identification of this quantities can imply oversmoothing or undersmoothing of estimated density. Some classical approaches for bandwidth selction are discussed in D. Ruppert (1995) and Fan and Yim (2004). Ad hoc method for selection of smoothing parameters proposed by Fan et al. (1996) has been adopted in this research to identify bandwidth h_r and h_z. This method consists of two steps. Firstly, bandwidth h_r has to be chosen based on the normal reference rule, see Silverman (1986). Afterwards, for a fixed bandwidth h_r and for a fixed r, (25) can be considered as a nonparametric regression of $K_h\left(R_\tau - r_\tau\right)$ on Z, where corresponding bandwidth h_z is selected using one-leave-out cross validation method.

Asymptotic distribution of the conditional physical density calculated using local linear regression has been derived in Song and Xiu (2016), and under conditions $1-6$ in the appendix of Aït-Sahalia et al. (2009) asymptotic distribution is given by

$$\begin{aligned} & k^{1/2}\left(h_r h_z\right)^{1/2}\left(\widehat{p}\left(r \mid \tau, z\right) - p\left(r \mid \tau, z\right)\right) \\ & \xrightarrow{\mathcal{L}} \mathrm{N}\left(0, \frac{p\left(r \mid \tau, z\right)}{\pi\left(z\right)} \left[\int K^2(c)dc\right]^3\right), \end{aligned} \tag{28}$$

as $kh_rh_z \to \infty$, where $\pi\left(z\right)$ is the density, k is a number of returns used in estimation procedure (see (24)) and K denotes a univariate kernel function as usually in the research.

4.2 Local constant regression for conditional density estimation

Classical kernel estimators of joint density $f(r,z)$ and marginal density $f(z)$ (see Härdle (2004)) have such a representation

$$\widehat{f}(r,z) = \frac{1}{n}\sum_{i=1}^{k} K_{h_r}(r-r_i)K_{h_z}(z-z_i) \tag{29}$$

$$\widehat{f}(z) = \frac{1}{n}\sum_{i=1}^{k} K_{h_z}(z-z_i) \tag{30}$$

As a consequence, conditional density $p(r \mid z)$ can be estimated as

$$\widehat{p}(r \mid z) = \frac{\widehat{f}(r,z)}{\widehat{f}(z)} = \frac{\sum_{i=1}^{k} K_{h_r}(r-r_i)K_{h_z}(z-z_i)}{\sum_{i=1}^{k} K_{h_z}(z-z_i)}. \tag{31}$$

It is known from the (25), that conditional density estimation boils down to nonparametric regression of $K_h(R_\tau - r_\tau)$ on Z. If one considers local constant regression of $K_h(R_\tau - r_\tau)$ on Z than the nonparametric kernel regression estimator $\widehat{m}_{NW}$ is then given by

$$\widehat{m}_{NW}(r \mid z) = \frac{1}{k}\sum_{i=1}^{k} K_{h_r}(r-r_i)\omega_i^{NW}(z), \tag{32}$$

where

$$\omega_i^{NW}(z) = \frac{K_{h_z}(z-z_i)}{\frac{1}{k}\sum_{i=1}^{k} K_{h_z}(z-z_i)}. \tag{33}$$

Thus, conditional density estimator obtained using nonparametric local constant kernel regression coincides with (31), and subscript NW in formula (33) is used to underline the fact that such an estimator is referred to as Nadaraya-Watson estimator. There are several approaches for bandwidth selection in case of Nadaraya-Watson estimator, see Härdle (2004) and Park and Marron (1990). Rule-of-thumb bandwidth for local constant regression proposed by Bowman and Azzalini (1997) has been used in this research.

As discussed in Li and Racine (2011) and Ruppert and Wand (1994), local constant regression has some advantages and disadvantages in comparison to local linear regression. Specifically, local linear estimator has a better bias performance and is design adaptive. Nevertheless, local constant estimator always provides non-negative estimates of conditional density function while local linear estimator can give negative estimates of conditional density.

To conclude discussion about use of local constant regression for estimation of conditional density, asymptotic distribution of this estimator must be presented. The formula for asymptotic variance of conditional physical density estimated based on Nadaraya-Watson estimator can be found in, e.g., Li and Racine (2011) and is given by:

$$
\mathrm{Avar}(\widehat{p}\,(r \mid z)) = \left[\int K^2(c)dc\right]^2 (kh_z h_r)^{-1} \widehat{p}\,(r \mid z)\, \pi\,(z)\,, \tag{34}
$$

as $kh_r h_z \to \infty$, where $\pi\,(z)$ is a marginal density, K is a kernel function, k is a number of returns used in estimation procedure (see (24)), h_z and h_r are bandwidths.

5 Empirical study

In this section, conditional pricing kernels as well as risk neutral and physical densities are estimated nonparametrically using options on the DAX 30 index, and empirical findings are discussed. Results presented in this section were obtained using the code that is published online at `www.github.com/QuantLet/pricing_kernels_and_implied_volatility` and `www.quantlet.de`. The data used in this empirical study can be provided by the author upon demand.

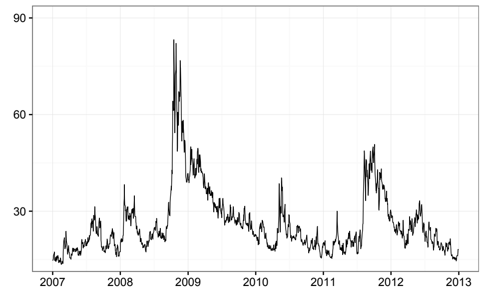
5.1 Data description

European call options on the DAX 30 index have been used for estimation of pricing kernels. This data is sourced by European Exchange EUREX, and has been taken from C.A.S.E., RDC SFB 649 (`sfb649.wiwi.hu-berlin.de`).

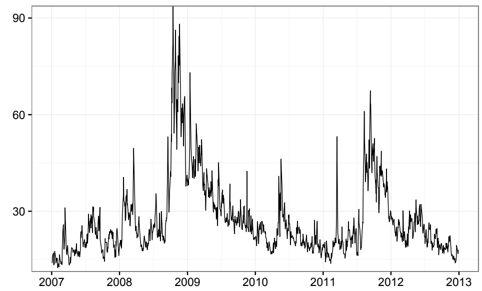
The implied volatility index VDAX-NEW has been considered as a proxy for volatility in this research. VDAX-NEW is a volatility index introduced by Deutsche Börse AG. This index expresses in percentage points the next 30 days implied volatility of the DAX 30 that is anticipated on the market. The volatility index VDAX-NEW is reported on an annual basis. The methodology for calculation of the VDAX-NEW has been developed by Deutsche Börse AG in cooperation with investment bank Goldman Sachs. Interpretation of VDAX-NEW is quite straightforward: high values of the index indicate huge amount of uncertainty on the market, while low values refer to market stability. Apart from VDAX-NEW, a set of subindexes corresponding to options with time to maturity from one month to two years are provided by Deutsche Boerse AG. Especially, its three subindexes: VDAX-NEW-Subindex 1, VDAX-NEW-Subindex 2 and VDAX-NEW-Subindex 3 corresponding to options with times to maturity 1, 2 and 3 months respectively have been used in this empirical study. Time series of VDAX-NEW subindeces have been downloaded from `www.onvista.de`.

The top panels of Figure 1 displays the time series of the DAX 30 Index and VDAX-NEW, while the bottom panels plot time series of the VDAX-NEW-Subindex 1, VDAX-NEW-Subindex 2 and VDAX-NEW-Subindex 3. Moreover, summary statistics of the DAX 30 index and volatility indexes are provided in Table 1. It can be observed from the top panels of Figure 1, that there is an obvious negative relation between the DAX 30 index and the volatility index. This fact is well-known in literature, see e.g., Carr and Wu (2009) and Schwert (1989), and can be used for portfolio diversification by adding a volatility security to portfolio.

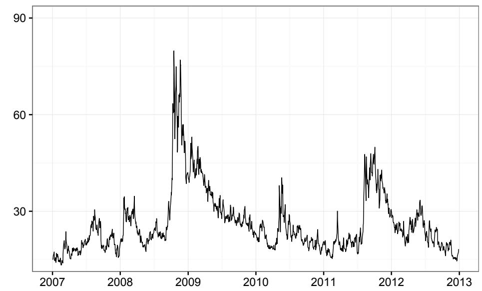
(a) DAX 30



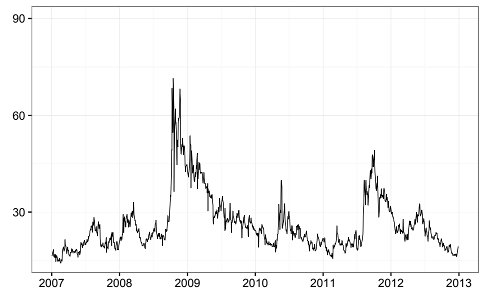
(b) VDAX-NEW



(c) VDAX-NEW-Subindex 1



(d) VDAX-NEW-Subindex 2



(e) VDAX-NEW-Subindex 3

Figure 1: Times series of the DAX 30 index and volatility indexes VDAX-NEW, VDAX-NEW-Subindex 1, VDAX-NEW-Subindex 2, VDAX-NEW-Subindex 3. Sample period is 1.01.2007 till 31.12.2012

Since option data is used for nonparametric estimation, it is necessary to perform some data-cleaning routines to get rid of redundant information that can add unnecessary complexity to the estimation procedure. The data-cleaning procedure often used in literature, e.g., Aït-Sahalia and Lo (1998), has been applied to option data. The option data contains call options with their characteristics as observed on the market, e.g., exercise price, settlement price, maturity and trading date. Only options with time to maturity between 7 days and

	DAX	VDAX	VDAX1m	VDAX2m	VDAX3m
Min.	3666.00	13.48	12.44	13.32	14.17
1st Qu.	5801.00	19.54	19.67	19.92	20.35
Median	6515.00	23.18	23.56	23.46	23.63
Mean	6389.00	26.16	27.00	26.28	26.16
3rd Qu.	7115.00	28.90	29.72	28.92	28.62
Max.	8106.00	83.23	95.11	79.80	71.44

Table 1: Summary statistics of time series of the DAX 30 index and volatility indexes VDAX-NEW, VDAX-NEW-Subindex 1, VDAX-NEW-Subindex 2, VDAX-NEW-Subindex 3. Sample period is 1.01.2007 till 31.12.2012

1 year have been considered. Additionally, the riskless interest rate has been determined for every option's time to maturity based on linear interpolation of EURIBOR.

5.2 Risk neutral and physical densities

As discussed in Section 2, estimation of risk neutral and physical densities is crucial for estimation of the pricing kernel. Estimates of these two densities, obtained based on the methodology discussed in Section 3 and Section 4 and available data, are discussed in this subsection.

Figure 2 provides nonparametric estimates of risk neutral and physical densities of market return conditional on different levels of VDAX-NEW. Risk neutral density has been estimated using local linear regression described in Section 3. Whereas physical density has been estimated based on local constant regression, and using the contemporaneous approach for estimation of the conditional physical density as discussed in Section 4. It can be seen that both risk-neutral and physical densities depend on volatility level. The densities are more spread-out if conditioned by a higher volatility level which corresponds to market instability. In other words, in times of market uncertainty (high volatility) both risk neutral and physical densities have fatter tails than in times of low market volatility. It can be also noticed that risk neutral density has a heavier left tail than physical density.

Figure 3 depicts estimates of physical and risk neutral densities with corresponding confidence intervals calculated using asymptotic theory presented in Section 3 and Section 4. Consistent with theory, confidence intervals for risk neutral density are narrower than for physical density. Also it can be noticed that tails of the both densities have narrower confi-

	year	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
1	2012	14.54	19.10	22.58	22.42	25.39	33.25
2	2011	15.31	19.23	22.20	27.58	37.15	50.74
3	2010	15.43	19.30	21.44	22.21	24.48	40.36
4	2009	22.11	27.06	30.83	32.75	38.48	50.00
5	2008	17.33	22.04	25.44	32.50	36.69	83.23
6	2007	13.48	16.83	18.66	19.42	21.41	31.42
7	2006	12.13	14.93	16.08	17.08	18.50	27.42
8	2005	11.65	13.09	14.09	14.57	15.78	19.30
9	2004	13.34	18.06	19.94	20.14	21.98	29.58
10	2003	22.18	27.04	31.54	34.66	43.75	56.71
11	2002	20.01	26.06	37.53	38.15	50.14	62.63

Table 2: Summary statistics of time series of the VDAX-NEW. Sample period is 1.01.2002 till 31.12.2012

dence intervals than middle points. This observation fully agrees with previous studies, e.g., Song and Xiu (2016).

Two approaches for estimation of the physical density have been presented in Section 4, i.e., local linear and local constant regression. Advantages and disadvantages of these methods have been mentioned in Section 3 and Section 4 and further details with regard to this issue can be found in Fan et al. (1996). One is enabled by Figure 4 to compare these two approaches for estimation of the physical density based on real data. Completely consistent with theory, Figure 4 shows that the local linear approach provides narrower confidence intervals for physical density than local constant kernel regression. It can also be seen from Figure 4 that for some range of returns the local linear approach fails to calculate estimated values of physical density, since the design matrix of the local linear regression has a more complicated structure than in the case of local constant regression. As a consequence, it is more likely to be singular. Additionally, the local linear approach can give negative estimates of conditional density. On the other hand, local constant regression for estimation of conditional density always provides non negative estimates.

		2012	2011	2010	2009	2008	2007	2006	2005	2004	2003	2002
time to maturity	Min.	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
	1st Qu.	0.14	0.14	0.15	0.16	0.14	0.14	0.15	0.15	0.13	0.12	0.12
	Median	0.28	0.31	0.33	0.35	0.30	0.29	0.32	0.33	0.25	0.25	0.27
	Mean	0.38	0.38	0.39	0.41	0.38	0.38	0.39	0.40	0.36	0.35	0.35
	3rd Qu.	0.62	0.62	0.62	0.65	0.61	0.60	0.62	0.64	0.56	0.55	0.56
	Max.	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
moneyness	Min.	0.07	0.07	0.07	0.08	0.10	0.20	0.27	0.13	0.14	0.25	0.30
	1st Qu.	0.76	0.79	0.73	0.73	0.84	0.77	0.78	0.79	0.78	0.81	0.88
	Median	0.93	0.96	0.92	0.97	1.02	0.92	0.92	0.93	0.96	1.02	1.06
	Mean	0.91	0.95	0.90	1.01	1.04	0.90	0.90	0.91	0.95	1.12	1.14
	3rd Qu.	1.09	1.13	1.07	1.23	1.22	1.05	1.04	1.05	1.12	1.31	1.35
	Max.	2.63	3.15	2.94	7.32	2.91	1.52	1.67	1.53	1.88	3.63	3.46

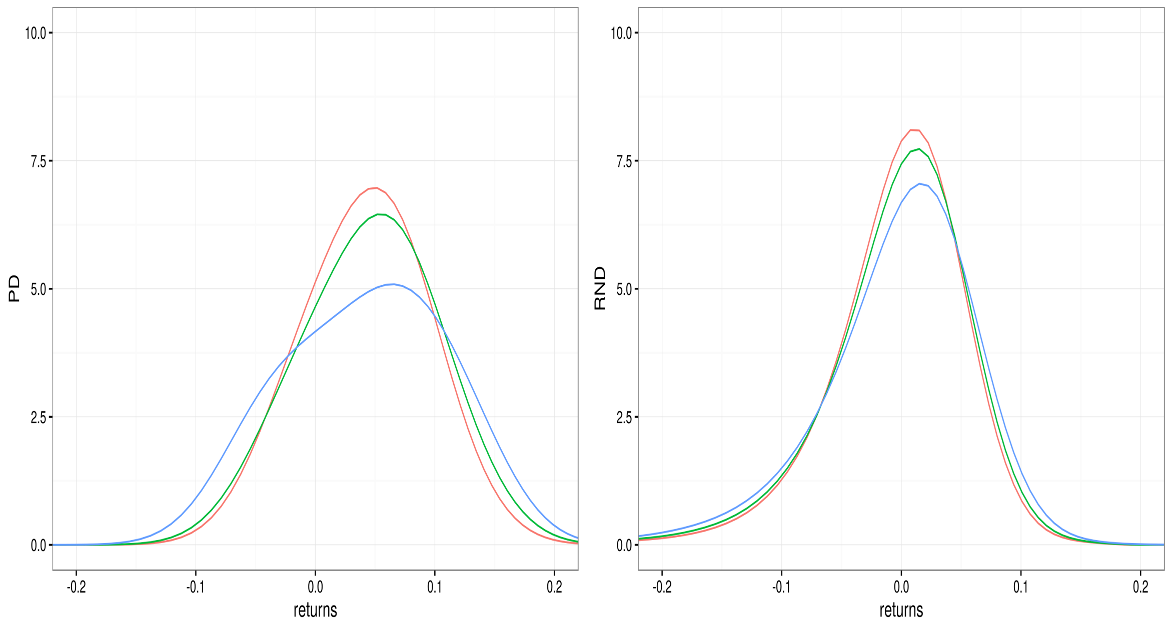
Table 3: Summary statistics of moneyness and time to maturity for selected DAX 30 options. Sample period is 1.01.2002 till 31.12.2012

5.3 Empirical pricing kernels

The main aim of this research is to investigate the dependence of the pricing kernels on implied volatility. Figure 5 depicts nonparametric estimates of the pricing kernel conditional on different values of implied volatility index.

Pricing kernel has been estimated based on the procedure described in Section 2, where risk neutral density has been estimated using local linear kernel regression and physical density has been estimated using local constant regression. It can be observed that depending on volatility pricing kernels can have different forms, which leads to the conclusion that volatility is an important factor in determining the shape of the pricing kernels.

With the purpose of further investigation of the relationship between pricing kernels and market volatility, a set of conditional empirical pricing kernels has been calculated for intervals of low, medium and high volatility as it is depicted in the left panels of Figure 9, which were obtained based on the following procedure. Firstly, a year of interest is specified (2012 in case of Figure 9) that determines the time span for data to be considered. Secondly, 5% and 95% quantiles of VDAX-NEW (or its subindex corresponding to the specified maturity) must be calculated. Then the sequence $v_i, i = 1, 2, \ldots, 30$ of equally spaced points from the 5%



(a) Physical density

(b) Risk neutral density

Figure 2: Physical and risk neutral densities of the DAX 30 index return conditional on time to maturity 1 month and on the 20% quantile (red curve), 40% quantile (green curve) and 60% quantile (blue curve) of VDAX-NEW. Sample period is 1.01.2012 till 31.12.2012

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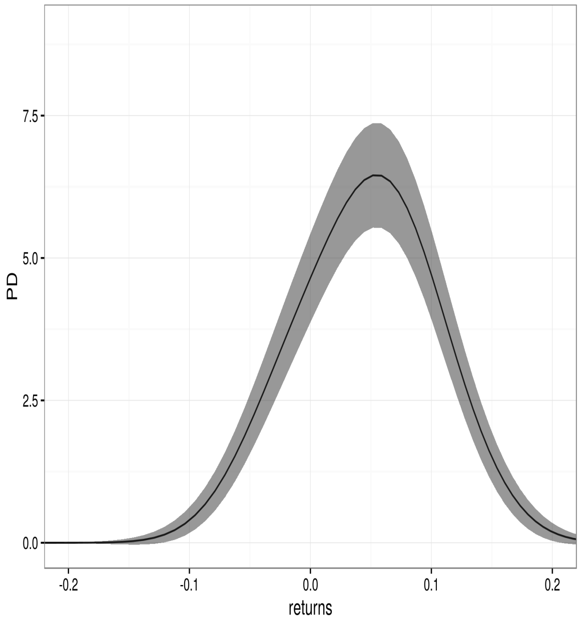
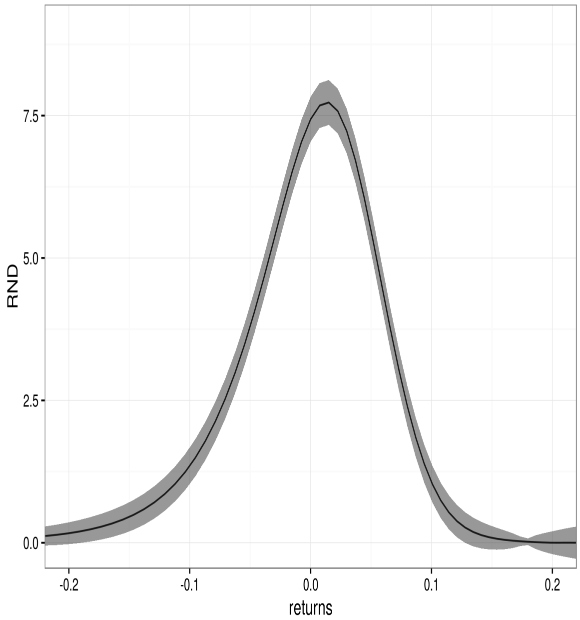
quantile to the 95% quantile of the volatility index were constructed, and all v_i have been grouped in intervals of low, high and medium volatility. Specifically, v_i less than 35% quantile belongs to low volatility, v_i with values between 35% and 65% quantile correspond to the medium volatility interval and v_i higher than 65% quantile are classified in the high volatility interval. Afterwards, conditional pricing kernels have been estimated conditioning on every v_i and specified time to maturity. The right-hand side of the Figure 9 depicts empirical pricing kernels conditional on 20%, 50% and 80% quantiles of the implied volatility index with 95% confidence intervals around each pricing kernel. Confidence intervals of pricing kernels have been calculated using the asymptotic theory of risk neutral and physical densities described in Section 2 and Section 3, and based on the formula for the asymptotic variance of pricing kernel proposed by Song and Xiu (2016)

$$\begin{aligned}\widehat{\text{Avar}}\left(\widehat{\mathcal{K}}\left(r \mid \tau, z\right)\right) = & \frac{1}{\left(\widehat{p}\left(r \mid \tau, z\right)\right)^2}\widehat{\text{Avar}}\left(\widehat{q}\left(r \mid \tau, z\right)\right) \\ & + \frac{\left(\widehat{q}\left(r \mid \tau, z\right)\right)^2}{\left(\widehat{p}\left(r \mid \tau, z\right)\right)^4}\widehat{\text{Avar}}\left(\widehat{p}\left(r \mid \tau, z\right)\right),\end{aligned} \tag{35}$$



(a) Physical density



(b) Risk neutral density

Figure 3: Physical and risk neutral density of the DAX 30 index return conditional on maturity 1 month and on the 40% quantile of VDAX-NEW. The grey areas are 95% confidence intervals. Sample period is 1.01.2012 till 31.12.2012

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where $\widehat{\text{Avar}}\left(\widehat{p}\left(r \mid \tau, z\right)\right)$ and $\widehat{\text{Avar}}\left(\widehat{q}\left(r \mid \tau, z\right)\right)$ are asymptotic variance of physical and risk neutral densities respectively.

Based on this procedure, pricing kernels conditional on different volatility intervals have been calculated, and are depicted in Figures 12 till 22 that can be found in the Appendix. It should be mentioned that colors from red to blue correspond to increasing values of volatility and all results are shown on a scale of continuously compounded 1-month period returns. From these figures we see that confidence intervals of pricing kernels are very wide for extreme returns. Therefore, we cannot draw any conclusion about the shape of pricing kernels in these areas.

Endpoints of intervals of low, medium and high volatility of VDAX-NEW and its subindexes are presented in Tables 7, 4, 5, 6 and in Figure 6.

Pricing kernels in the low volatility interval of year 2009 (with endpoints 23.80 and 27.97) are strictly decreasing, whereas pricing kernels in the low (endpoints 16.76 and 20.20) and medium (endpoints 20.20 and 24.20) volatility intervals of year 2012 are hump-shaped. This means that in times of high volatility pricing kernels tend to be decreasing which corresponds

		low		medium		high	
	year	from	to	from	to	from	to
1	2012	15.88	20.25	20.25	24.17	24.17	29.18
2	2011	15.80	20.54	20.54	35.17	35.17	51.05
3	2010	17.21	19.98	19.98	23.08	23.08	29.26
4	2009	23.98	28.15	28.15	35.78	35.78	49.16
5	2008	18.92	23.32	23.32	32.34	32.34	73.63
6	2007	13.71	17.58	17.58	21.24	21.24	28.21

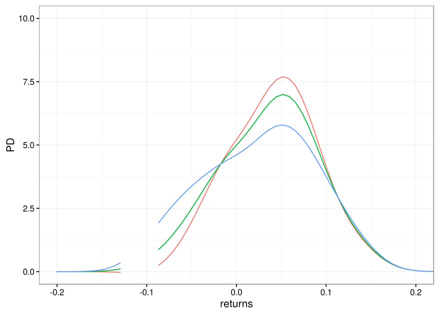
Table 4: Endpoints of the volatility intervals, VDAX-NEW-Subindex 1

		low		medium		high	
	year	from	to	from	to	from	to
1	2012	15.61	20.39	20.39	24.56	24.56	29.46
2	2011	16.70	20.40	20.40	32.39	32.39	44.96
3	2010	18.16	20.55	20.55	23.00	23.00	29.36
4	2009	24.38	28.13	28.13	34.96	34.96	46.21
5	2008	19.82	23.02	23.02	29.41	29.41	66.04
6	2007	14.67	18.01	18.01	20.26	20.26	26.32

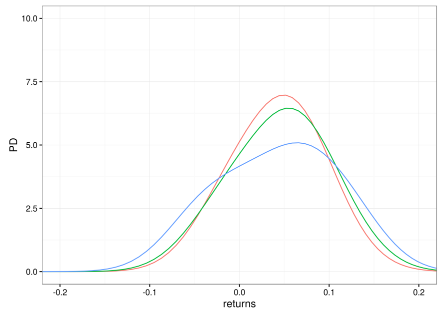
Table 5: Endpoints of the volatility intervals, VDAX-NEW-Subindex 2

		low		medium		high	
	year	from	to	from	to	from	to
1	2012	16.84	21.80	21.80	24.68	24.68	29.74
2	2011	17.22	20.47	20.47	31.92	31.92	43.22
3	2010	18.70	21.19	21.19	23.57	23.57	29.49
4	2009	24.91	28.03	28.03	34.57	34.57	44.74
5	2008	20.30	23.34	23.34	27.94	27.94	59.19
6	2007	15.07	18.37	18.37	20.51	20.51	25.52

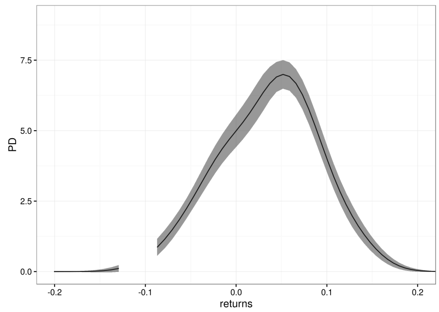
Table 6: Endpoints of the volatility intervals, VDAX-NEW-Subindex 3



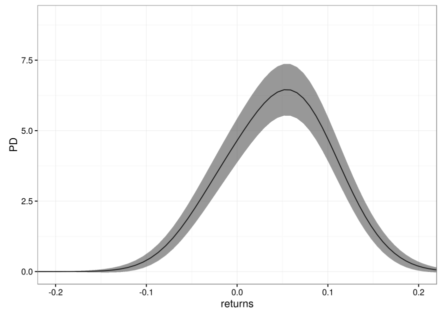
(a) PDF estimated using local linear regression.



(b) PDF estimated using local constant regression.



(c) PDF estimated using local linear regression.



(d) PDF estimated using local constant regression.

Figure 4: Comparison of local constant and local linear regressions for estimation of physical density. Note: The top panels provide nonparametric estimates of physical density of the DAX 30 index return conditional on time to maturity 1 month and on the 20% quantile (red curve), 40% quantile (green curve) and 60% quantile (blue curve) of VDAX-NEW. The bottom panels provide nonparametric estimates of physical density conditional on time to maturity 1 month and on the 40% quantile of VDAX-NEW. The grey areas are 95% confidence intervals. Sample period is 1.01.2012 till 31.12.2012.

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to risk-averse behavior of investors, but in times of low market volatility the hump of the empirical pricing kernel is more noticeable. This results is consistent with Grith et al. (2016) and Grith et al. (2013). A natural interpretation for this is that investors do not prefer risky investments in times of high market uncertainty.

There are three well-known patterns of empirical pricing kernels. Specifically, pricing kernels can be decreasing, U-shaped or have a hump near zero returns. Although we are certain about the fact that volatility is not a unique state variable driving the shape of the pricing kernel, we are going to investigate how volatility influences the shape of pricing kernel. In other words, the link between the shape of pricing kernel and volatility regime has to be

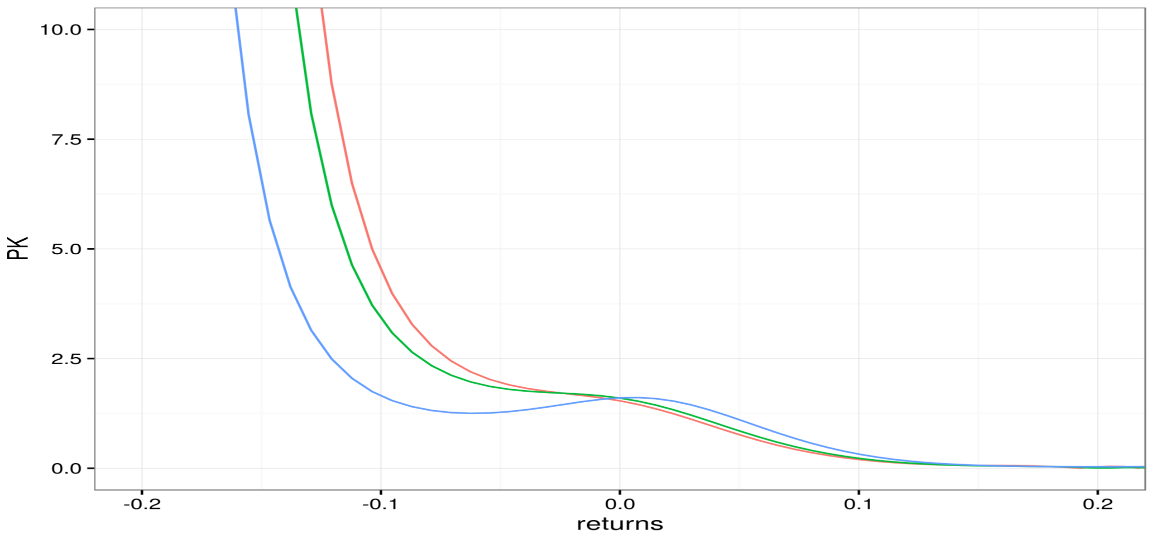

Figure 5: Pricing kernel conditional on time to maturity one month and different levels of VDAX-NEW: the 20% quantile (18.81) of VDAX-NEW corresponds to a red curve, 40% quantile (20.71) to a green curve and 60% quantile (23.40) to a blue curve. Sample period is 1.01.2012 till 31.12.2012. Note: corresponding risk neutral and physical densities are depicted in Figure 2.

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established.

Based on estimated pricing kernels for the years from 2002 till 2012, conditional on time to maturity of one months and levels of VDAX-NEW, the following conclusions can be drawn. Firstly, one can observe that in times of very law volatility (low volatility intervals of 2007, 2006, 2004 and the whole year 2005) pricing kernels tend to be U-shaped or decreasing, see Figures 17, 18, 20 and 19. Confidence intervals on the edges are very broad, that is why we can not determine if the pricing kernel is U-shaped or decreasing. Thus, one can conclude that pricing kernels possess a U-shaped (or decreasing) form if conditioned by volatility levels less than 18. Moreover, pricing kernels are mostly hump-shaped if conditioned by volatility from 18 to 25. It can be also noticed that the pricing kernel became decreasing if conditioned by volatility levels bigger then 25, and this pattern holds till volatility level of 35. For higher volatility pricing kernels have very wide confidence intervals which leads to inability to judge their shape. Nevertheless, it should be mentioned that pricing kernels possess patterns specific for each year. For instance, years 2003 and 2002 have comparable volatility ranges, but they have different shapes of pricing kernels.

Since pricing kernels are constructed as a ratio of risk neutral and physical densities,

		low		medium		high	
	year	from	to	from	to	from	to
1	2012	15.69	20.20	20.20	24.20	24.20	29.31
2	2011	16.54	20.37	20.37	32.12	32.12	46.06
3	2010	17.73	20.10	20.10	22.64	22.64	29.37
4	2009	23.80	27.97	27.97	34.93	34.93	46.33
5	2008	19.21	22.99	22.99	29.81	29.81	65.64
6	2007	14.59	17.77	17.77	20.02	20.02	26.51
7	2006	13.70	15.33	15.33	17.16	17.16	23.96
8	2005	12.47	13.53	13.53	15.04	15.04	18.17
9	2004	15.56	19.01	19.01	20.87	20.87	25.88
10	2003	24.23	28.81	28.81	36.00	36.00	51.12
11	2002	22.61	28.25	28.25	46.65	46.65	58.05

Table 7: Endpoints of the volatility intervals, VDAX-NEW

understanding the relationship between these densities and implied volatility can shed light on the connection between shape of pricing kernel and volatility regime.

Let us take a look at the physical and risk neutral densities used for estimation of the pricing kernels shown in Figure 14. Figure 7 depicts risk neutral and physical densities conditional on different volatility intervals. It can be observed that risk neutral densities have a negative skew (the probability mass is concentrated on the right side. In other words, it is a left-skewed distribution) for all volatility intervals, whereas physical density tends to have a positive skew (the probability mass is concentrated on the left side. In other words, it is a right-skewed distribution) and this tendency becomes more pronounced if volatility increases.

As observed in Figure 14, pricing kernels in year 2010 have a U-shaped form. As a consequence, one can conclude that the combination of left-skewed risk neutral density and right-skewed physical density provide a U-shaped pricing kernel. In the same fashion years 2002-2012 were analyzed, and it can be concluded that the combination of right-skewed (or symmetric) risk neutral density and left-skewed physical density leads to a decreasing pricing kernel. What's more, left-skewed (or symmetric) risk neutral density in conjunction with right-skewed physical density provides a hump-shaped or U-shaped pricing kernel. Although years 2011, 2008 and 2006 are exceptions to this rule.

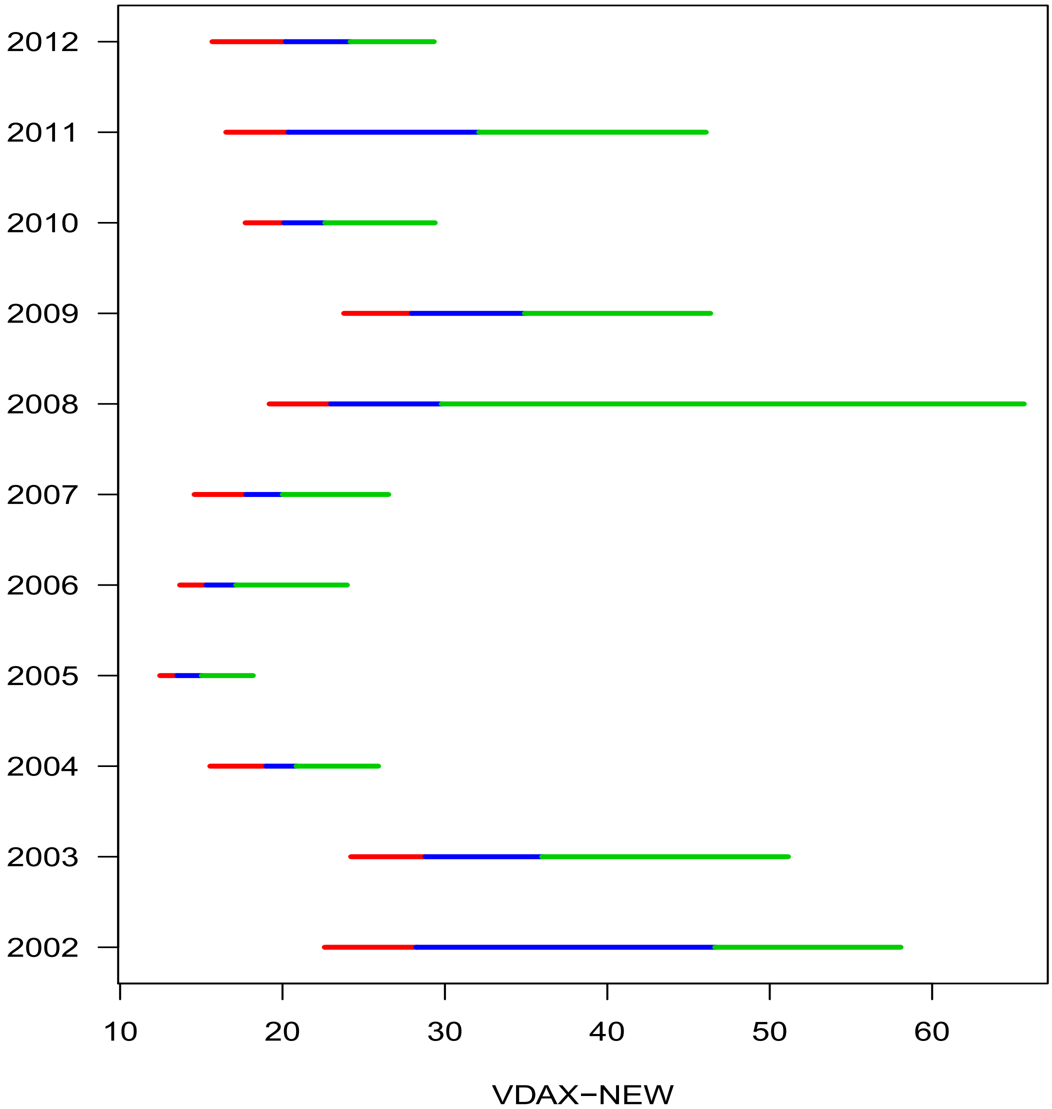

Figure 6: Intervals of law(red color), medium(blue color) and high(green color) volatility for the years from 2002 till 2012, VDAX-NEW

Time to maturity can also influence the shape of pricing kernels conditional on a given volatility level. Figures 9, 10 and 11 depict empirical pricing kernels calculated for the year 2012 conditional on times to maturity of 1, 2 and 3 months respectively. To make the comparison of pricing kernels conditional on different maturities more concrete let us consider Figure 8 that depicts empirical pricing kernels conditional on times to maturity of 1, 2 and 3 months and different levels of VDAX-NEW (10 equally spaced numbers from 35% to 65%

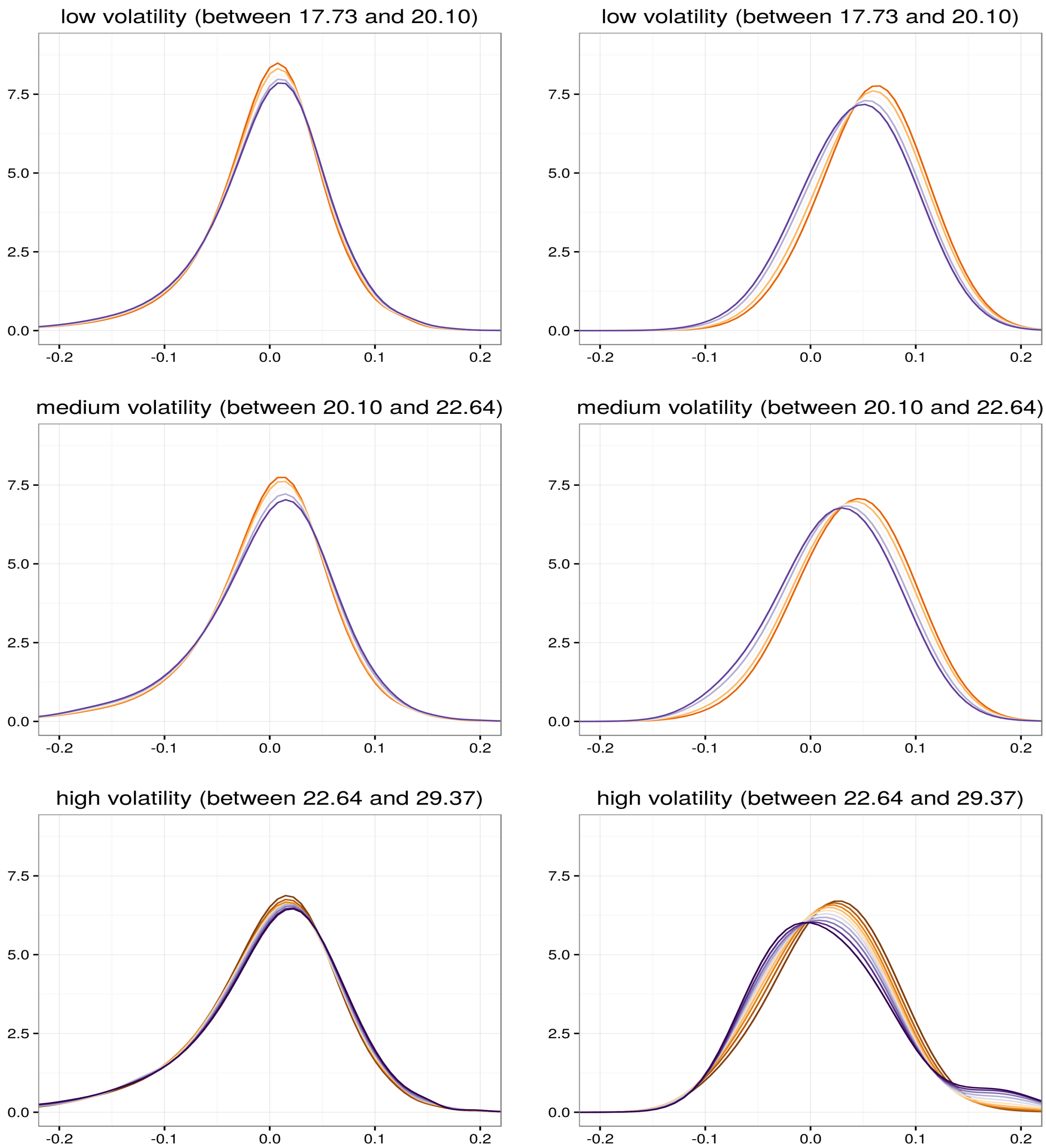

Figure 7: Risk neutral (left panels) and physical (right panels) densities conditional on time to maturity 1 month and on different values of VDAX-NEW. Sample period is 1.01.2010 till 31.12.2010. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 7. Corresponding pricing kernels are depicted in Figure 14.

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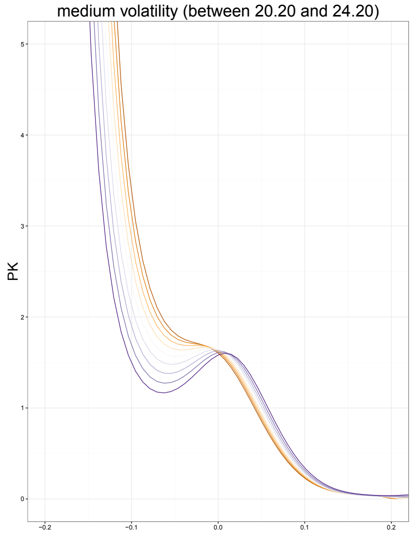
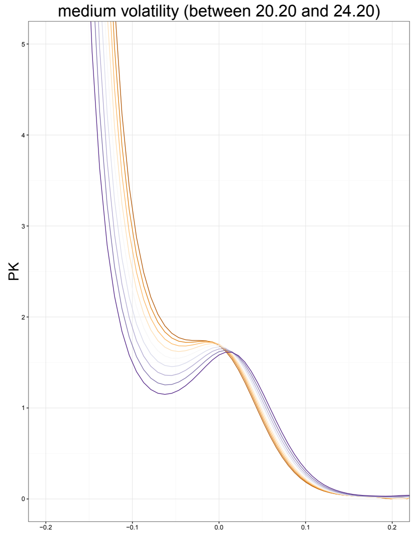
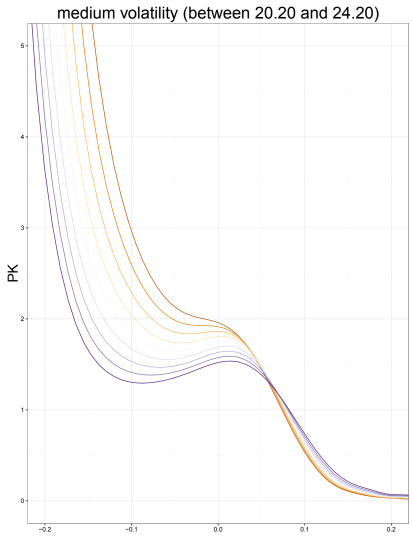
quantile of VDAX-NEW in 2012). VDAX-NEW (and VDAX-NEW-Subindex 1), VDAX-NEW-Subindex 2 and VDAX-NEW-Subindex 3 were used for estimation of pricing kernels



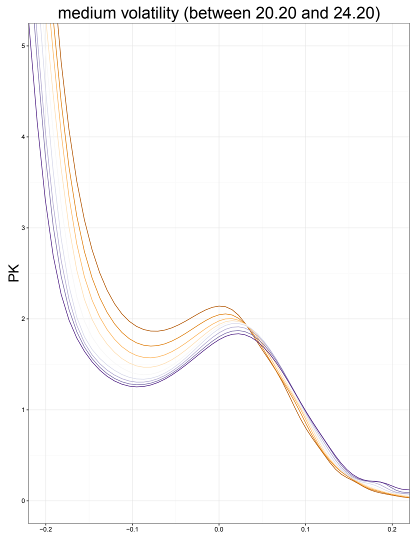
(a) 1 month maturity, VDAX-NEW



(b) 1 month maturity, VDAX-NEW-Subindex 1



(c) 2 months maturity, VDAX-NEW-Subindex 2



(d) 3 months maturity, VDAX-NEW-Subindex 3

Figure 8: Term structure of conditional empirical pricing kernels. Sample period is 1.01.2012 till 31.12.2012. Note: Colors from red to blue correspond to increasing values of volatility within each interval.

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conditioned by times to maturity of 1, 2 and 3 months respectively. Based on these figures it can be concluded that the spread of the hump of pricing kernels becomes bigger with longer time to maturity. In other words, if time to maturity increases then the hump of the pricing kernel becomes wider. Such a relationship between the pricing kernel and time to maturity is typical for all years from 2002 till 2012.

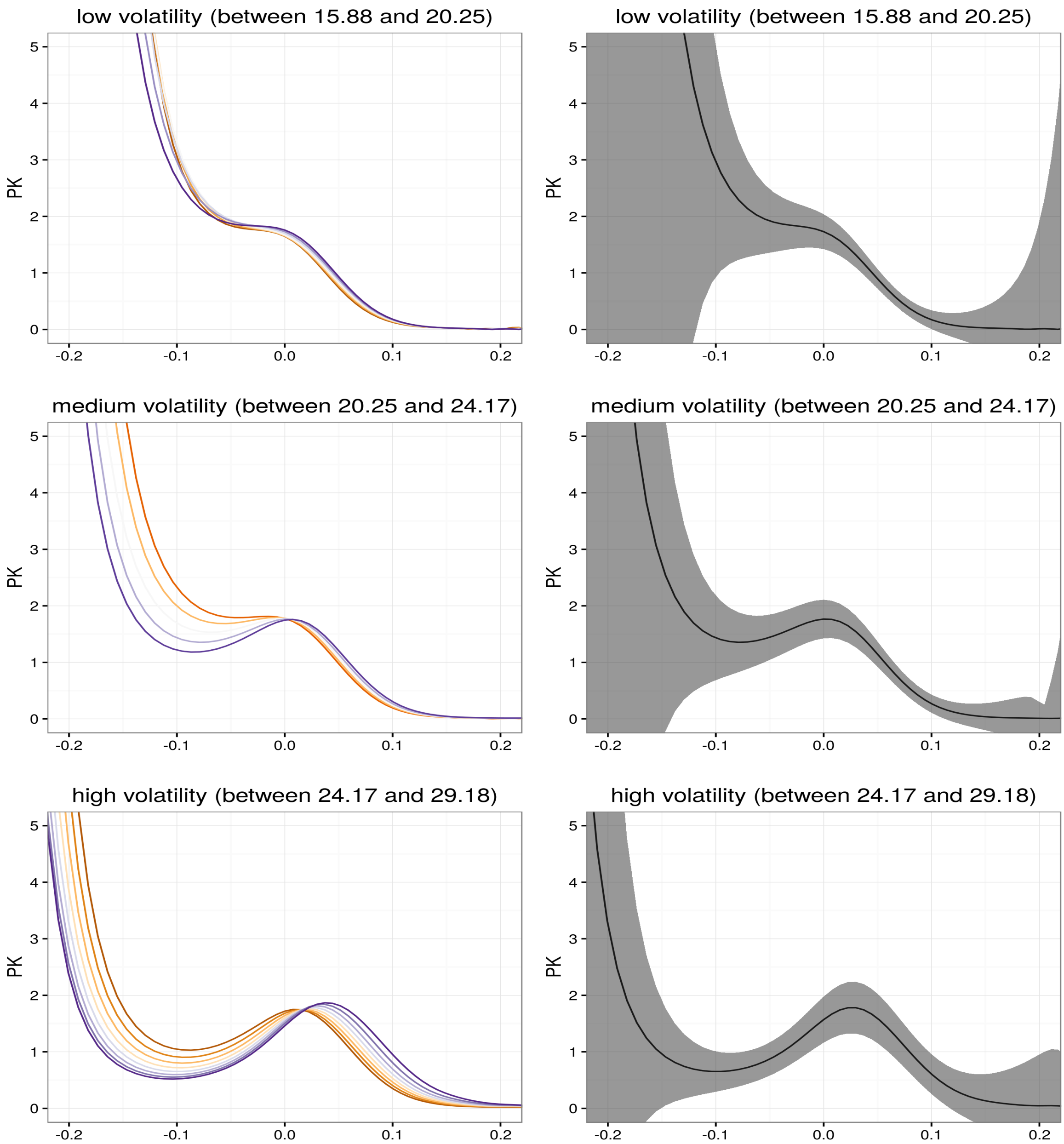

Figure 9: Pricing kernels conditional on time to maturity 1 month and on different values of VDAX-NEW-Subindex 1. Sample period is 1.01.2012 till 31.12.2012. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 4. The panels on the right-hand side depict empirical pricing kernels conditional on 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

`epk3VolaIntervalsVDAX1m`

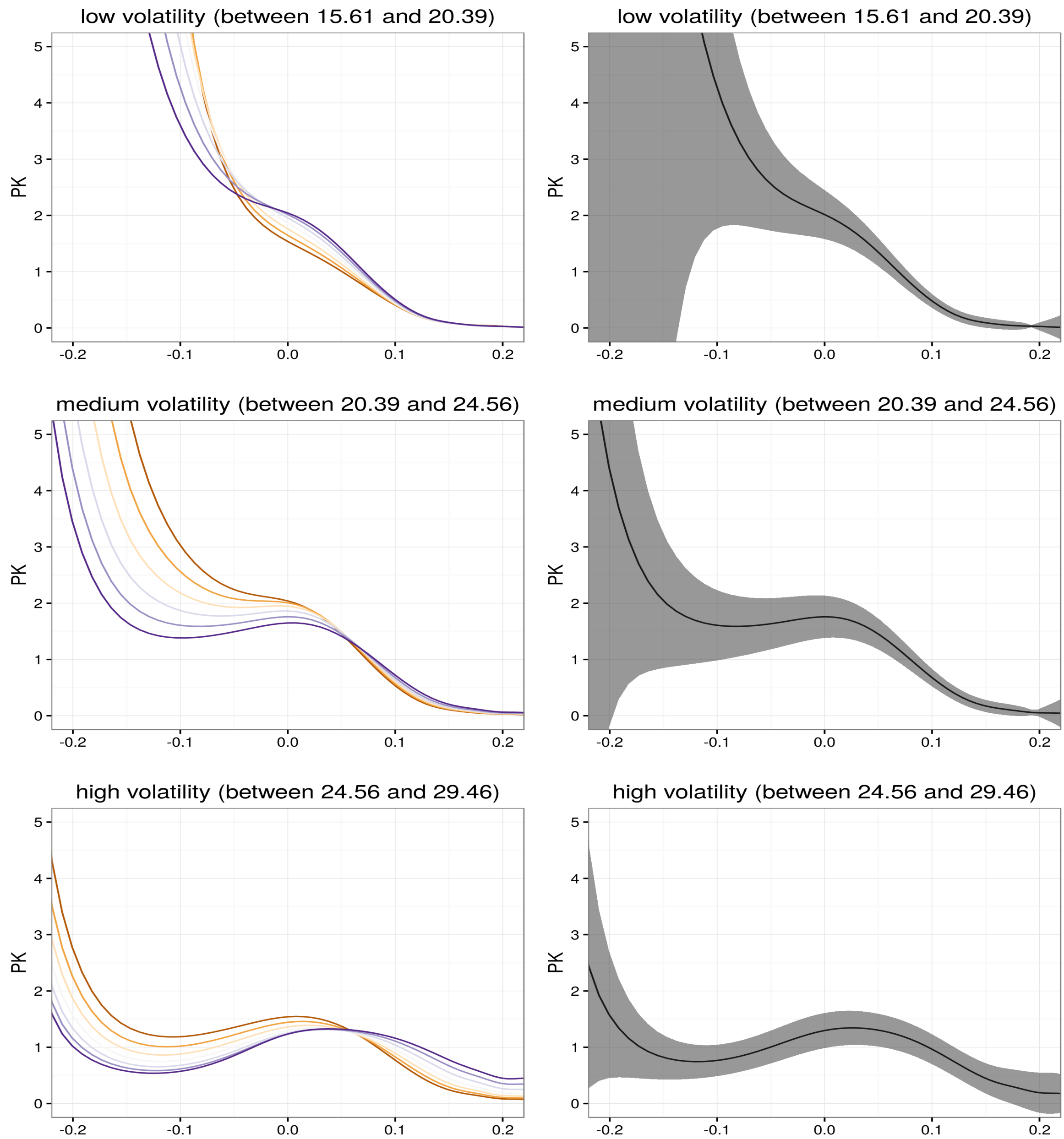

Figure 10: Pricing kernels conditional on time to maturity 2 months and on different values of VDAX-NEW-Subindex 2. Sample period is 1.01.2012 till 31.12.2012. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 5. The panels on the right-hand side depict empirical pricing kernels conditional on 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

`epk3VolaIntervalsVDAX2m`

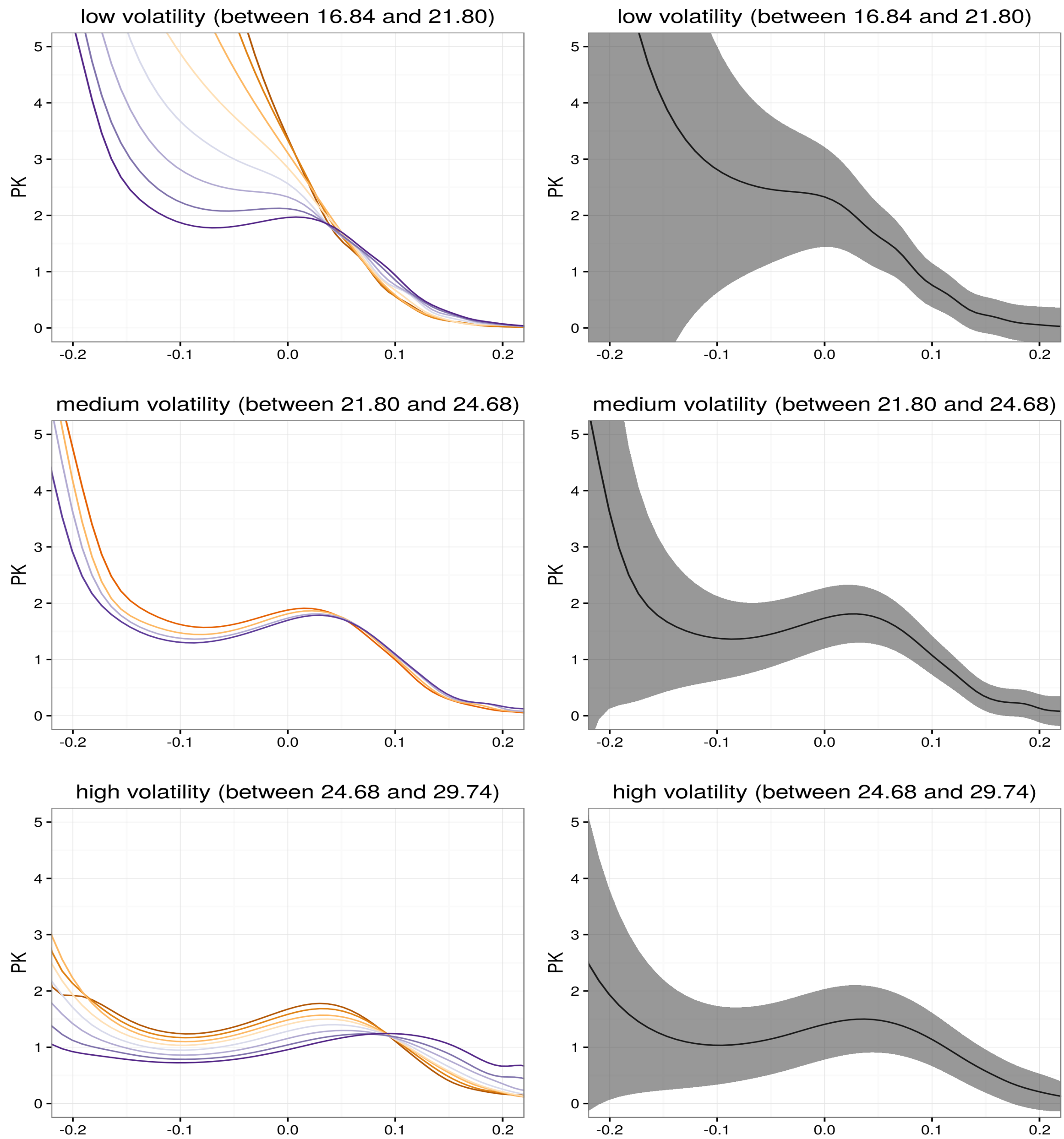

Figure 11: Pricing kernel conditional on time to maturity 3 months and on different values of VDAX-NEW-Subindex 3. Sample period is 1.01.2012 till 31.12.2012. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 6. The panels on the right-hand side depict empirical pricing kernels conditional on 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

epk3VolaIntervalsVDAX3m

6 Conclusions

Pricing kernels are crucial elements in understanding investor perception of market risk. The implied volatility index was considered in this thesis as a state variable that can explain changes in the shape of pricing kernels. The methodology for estimation of conditional pricing kernels is presented in this thesis. Moreover, two nonparametric techniques for estimation of conditional physical density were compared. Yearly conditional pricing kernels for the time from 2002 till 2012 were estimated using empirical data.

Obtained estimates show how the shape of the pricing kernels depends on volatility level. Specifically, it was observed that conditioning on low levels of implied volatility index VDAX-NEW (less than 18) provides U-shaped empirical pricing kernels. Furthermore, conditioning on relatively medium volatility levels (from 18 to 25) leads to mostly hump-shaped pricing kernels, and conditioning on high volatility (from 25 to 35) results in decreasing pricing kernels. The last result suggests that in times of high market volatility investors do not want to invest in risky securities, in other words, they exhibit risk-averse behavior, which is reflected by a concave utility function, and, as a consequence, implies monotonically decreasing pricing kernels. Although 2011, 2006 and 2002 are exceptions to this rule. Conditioning on extremely high volatility levels (higher than 35) leads to mostly U-shaped pricing kernels. However, pricing kernels in this case have very wide confidence intervals, which leads to inability to make any conclusion about the form of the pricing kernel.

Furthermore, the influence of different times to maturity on the shape of pricing kernels was investigated. For such a purpose, subindexes of VDAX-NEW corresponding to different maturities were employed. It was noticed that if time to maturity increases then the hump of pricing kernel becomes wider.

In addition, skewness of risk neutral and physical densities was related to the shape of the pricing kernels. Notably, it was observed that the combination of right-skewed (or symmetric) risk neutral density and left-skewed physical density leads to a decreasing pricing kernel, whereas left-skewed (or symmetric) risk neutral density together with right-skewed physical density provides a hump-shaped or U-shaped pricing kernel. Although years 2011, 2008 and 2006 are exceptions to this rule.

7 Appendix

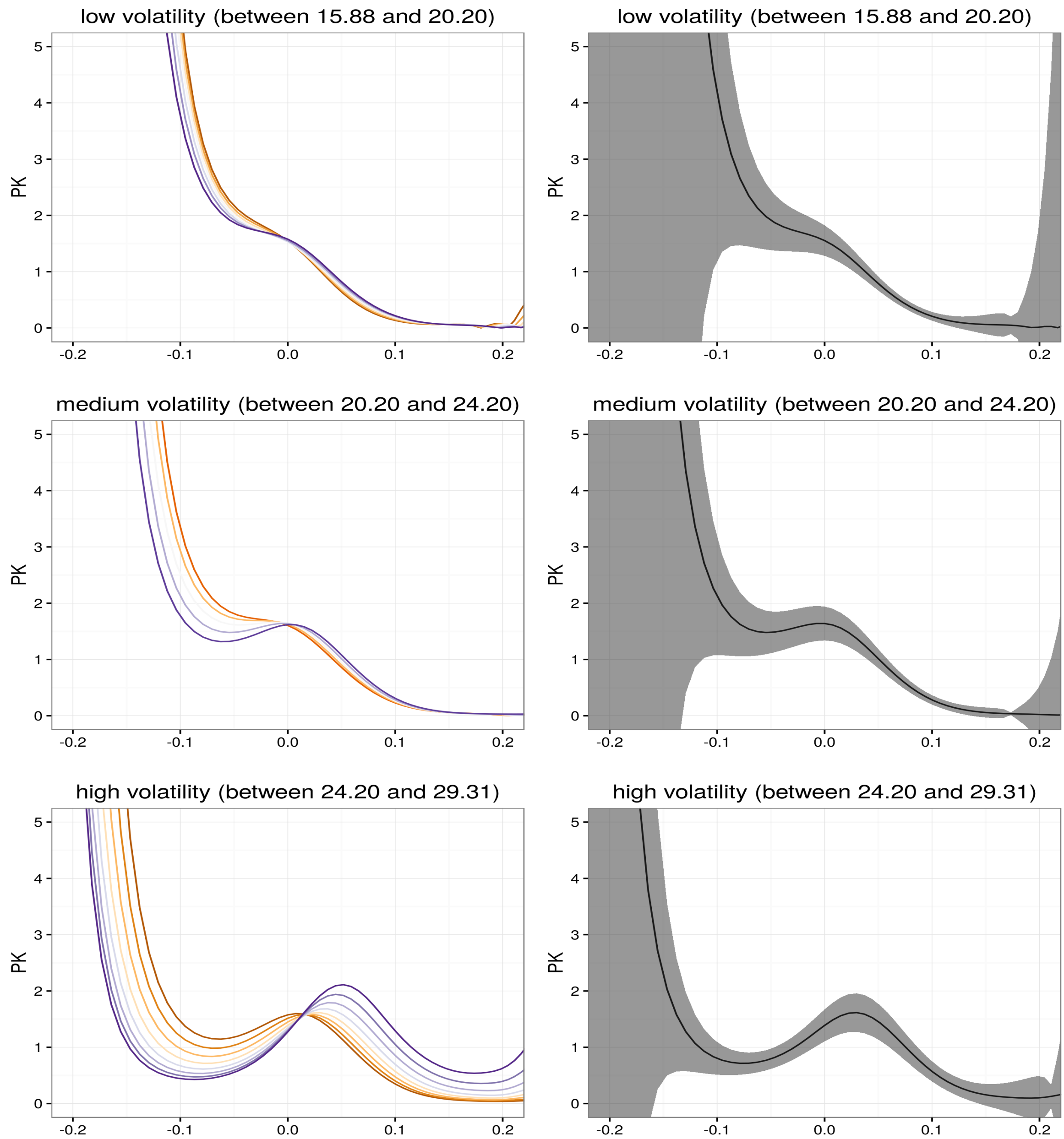

Figure 12: Pricing kernels conditional on time to maturity one month and on different values of VDAX-NEW. Sample period is 1.01.2012 till 31.12.2012. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 7. The panels on the right-hand side depict empirical pricing kernels conditional on the 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

epk3VolaIntervalsVDAX

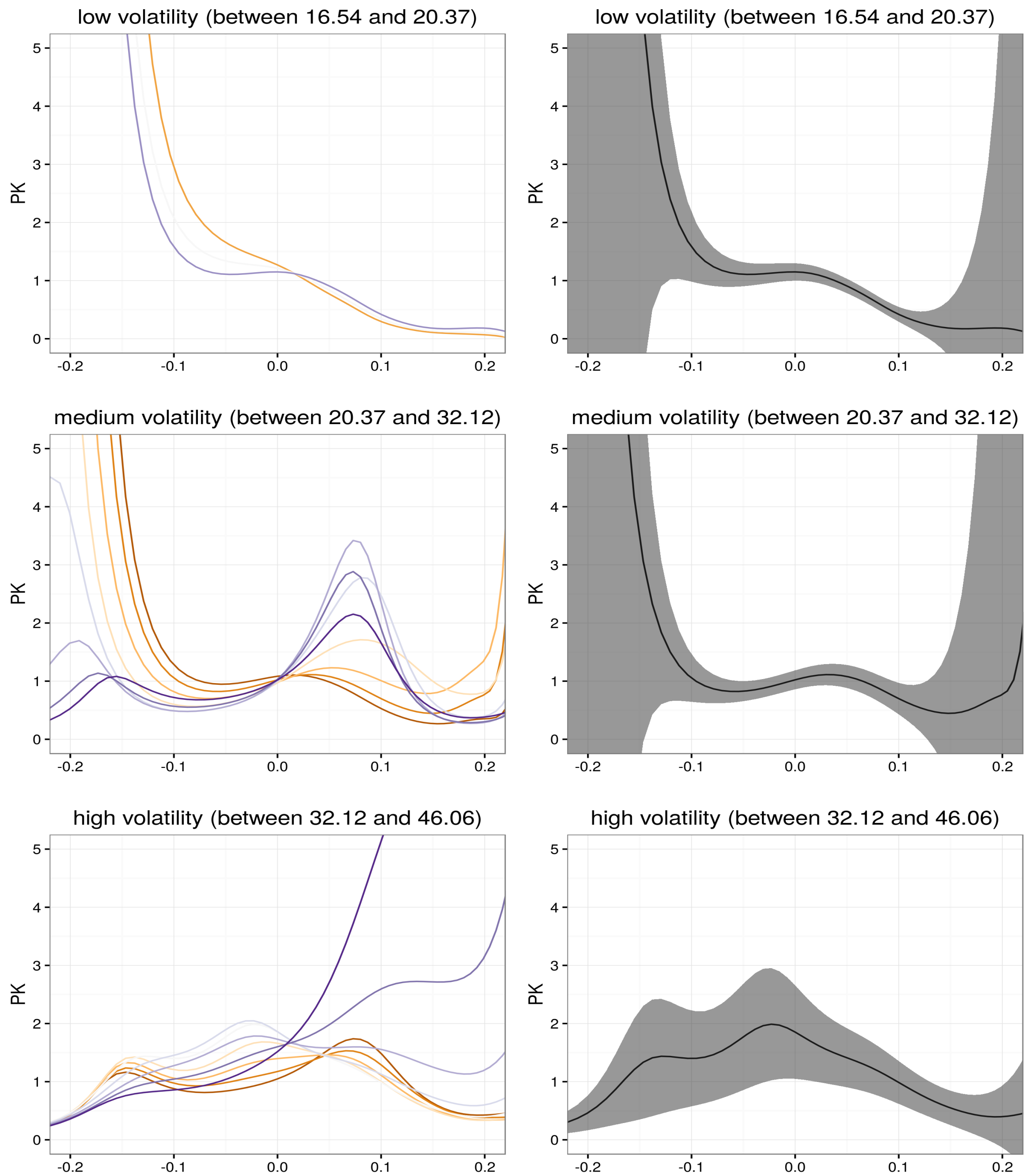

Figure 13: Pricing kernels conditional on time to maturity one month and on different values of VDAX-NEW. Sample period is 1.01.2011 till 31.12.2011. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 7. The panels on the right-hand side depict empirical pricing kernels conditional on the 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

epk3VolaIntervalsVDAX

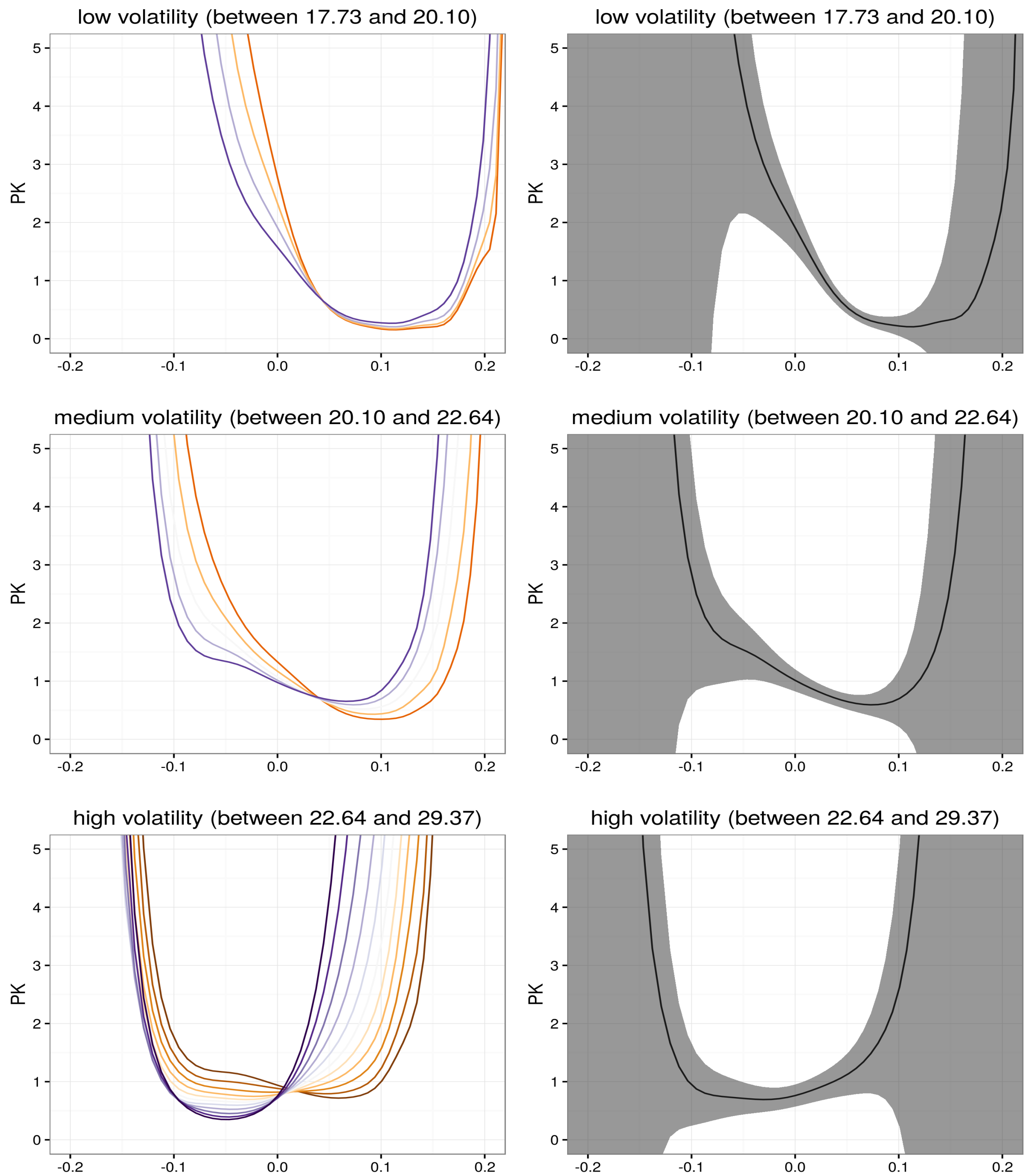

Figure 14: Pricing kernels conditional on time to maturity one month and on different values of VDAX-NEW. Sample period is 1.01.2010 till 31.12.2010. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 7. The panels on the right-hand side depict empirical pricing kernels conditional on the 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

epk3VolaIntervalsVDAX

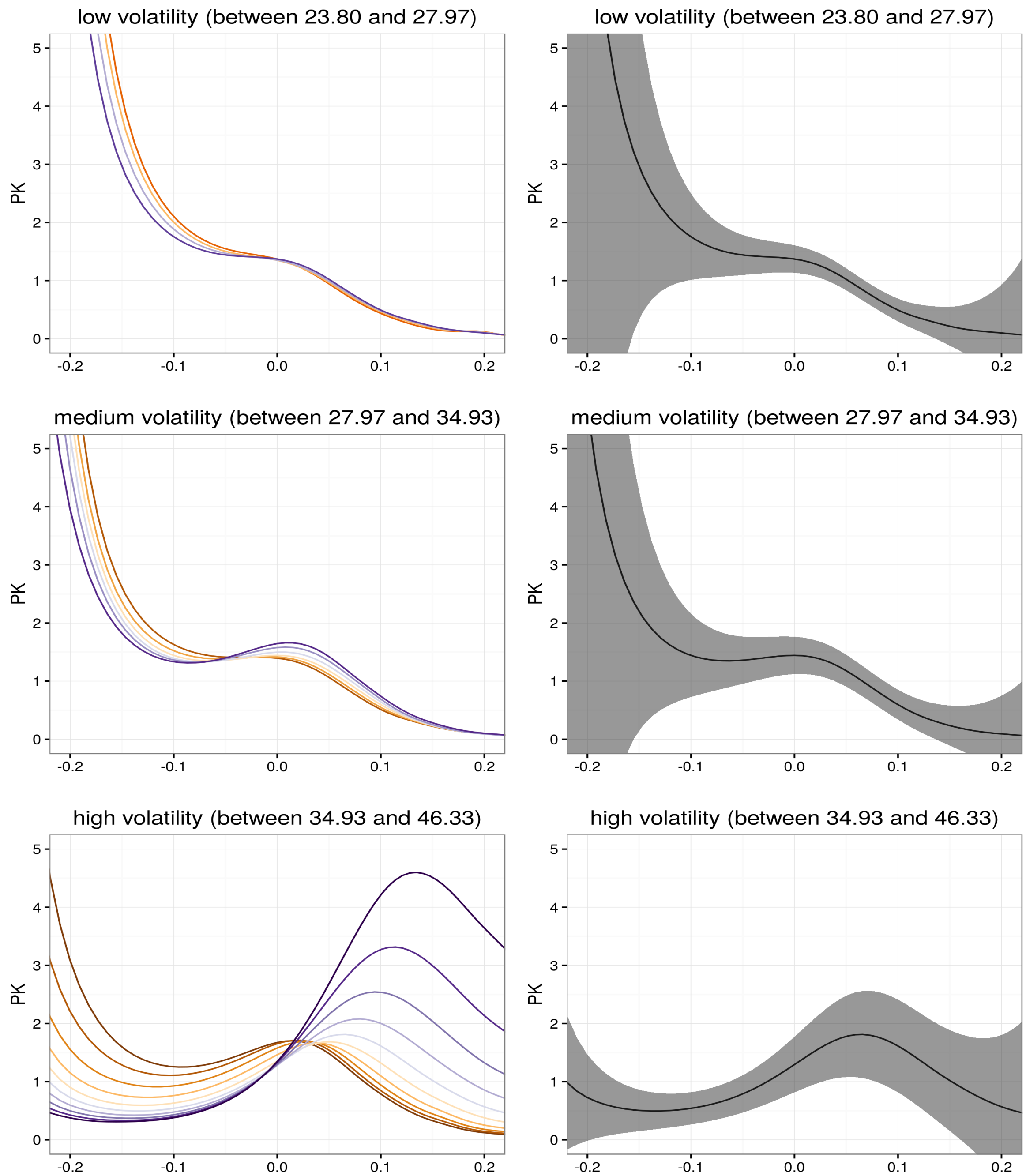

Figure 15: Pricing kernels conditional on time to maturity one month and on different values of VDAX-NEW. Sample period is 1.01.2009 till 31.12.2009. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 7. The panels on the right-hand side depict empirical pricing kernels conditional on the 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

epk3VolaIntervalsVDAX

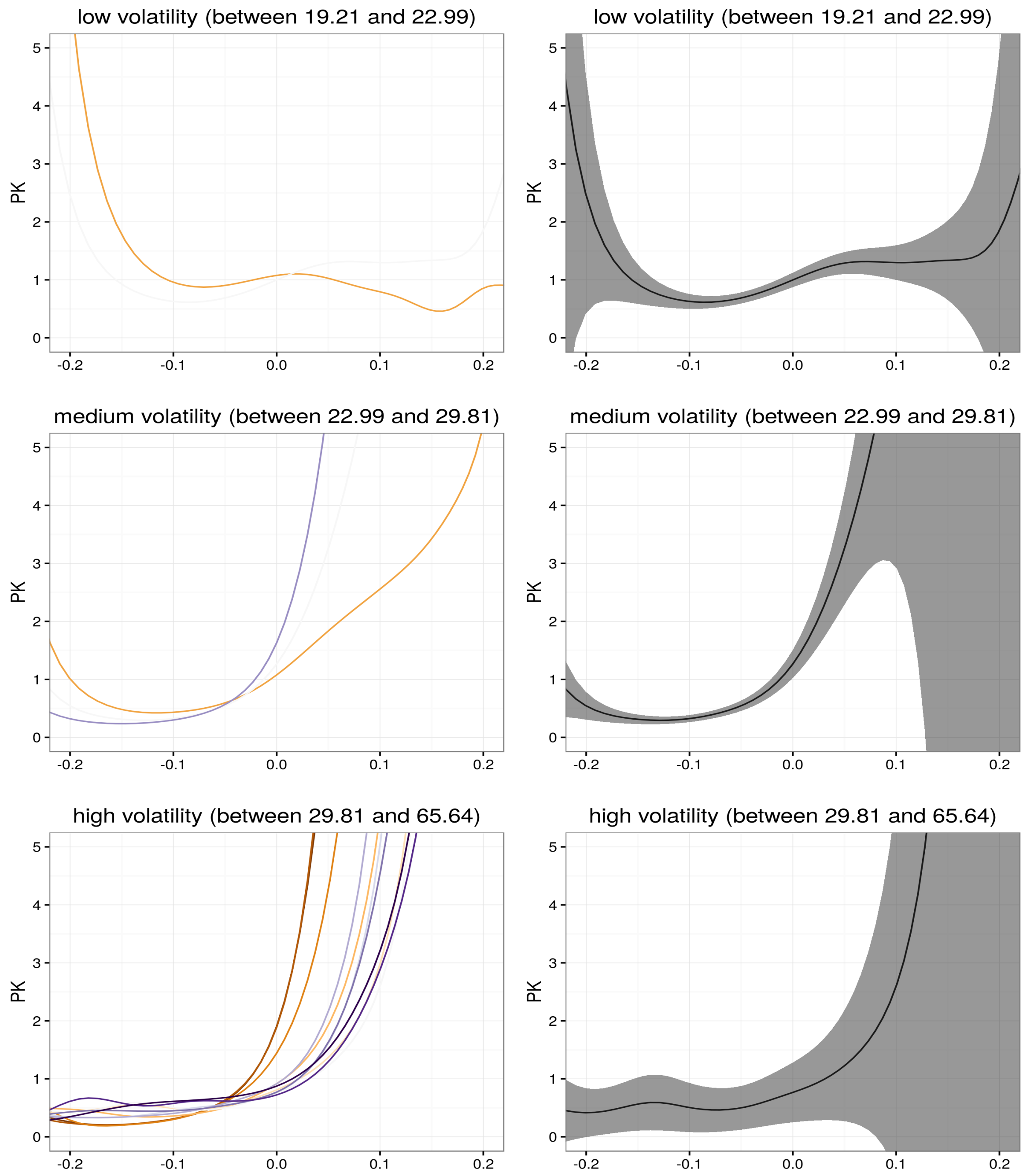

Figure 16: Pricing kernels conditional on time to maturity one month and on different values of VDAX-NEW. Sample period is 1.01.2008 till 31.12.2008. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 7. The panels on the right-hand side depict empirical pricing kernels conditional on the 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

epk3VolaIntervalsVDAX

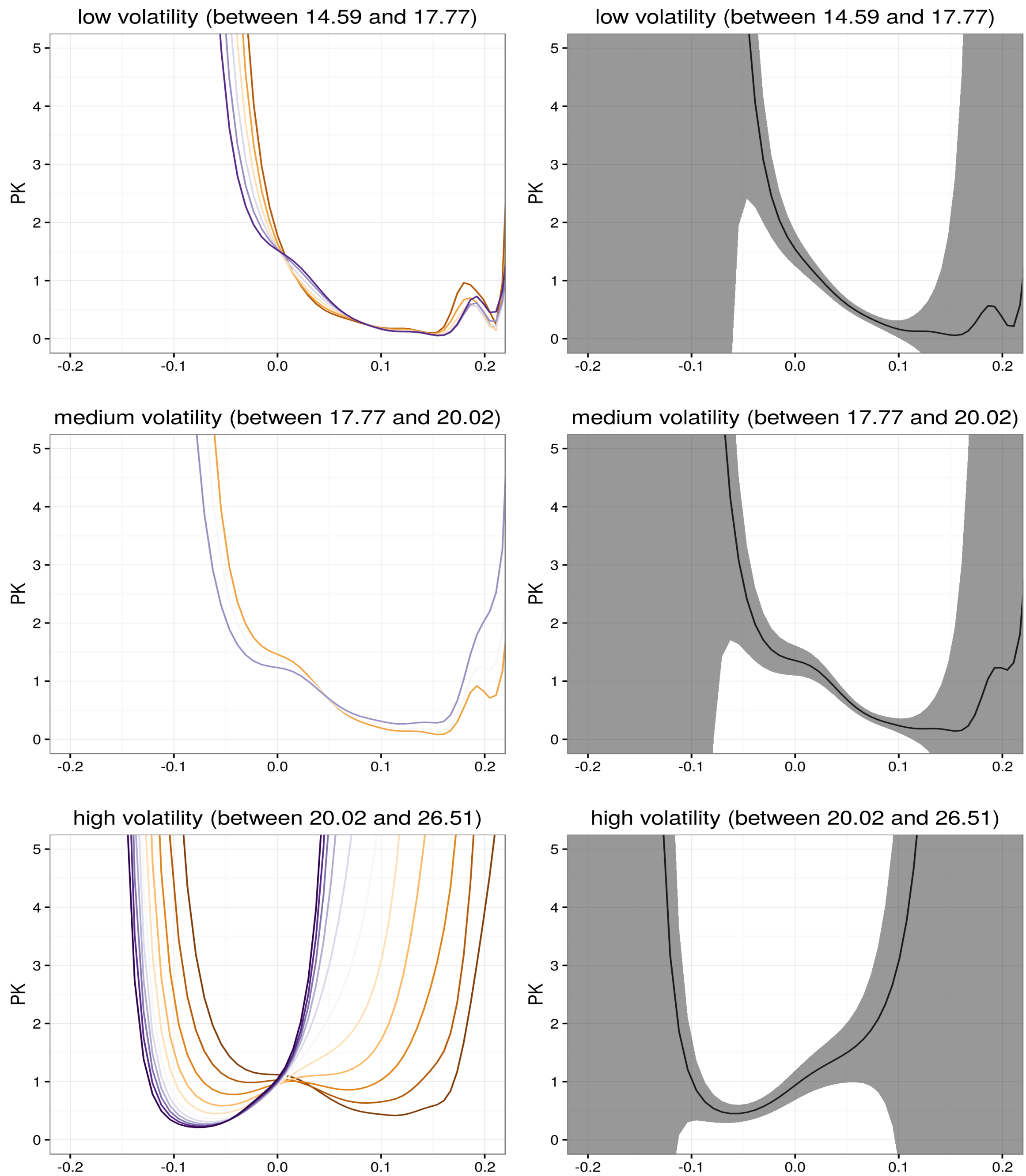

Figure 17: Pricing kernels conditional on time to maturity one month and on different values of VDAX-NEW. Sample period is 1.01.2007 till 31.12.2007. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 7. The panels on the right-hand side depict empirical pricing kernels conditional on the 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

epk3VolaIntervalsVDAX

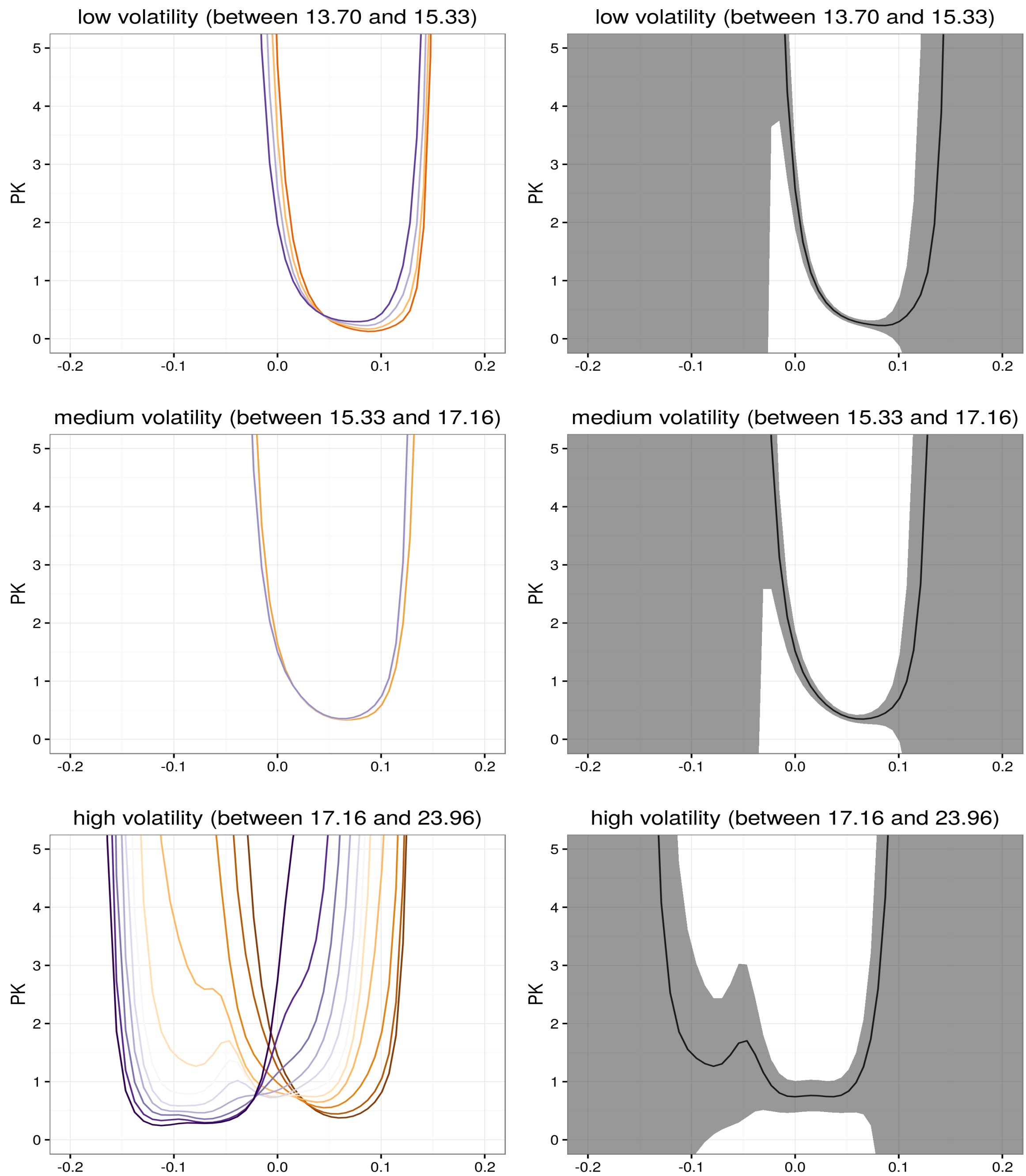

Figure 18: Pricing kernels conditional on time to maturity one month and on different values of VDAX-NEW. Sample period is 1.01.2006 till 31.12.2006. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 7. The panels on the right-hand side depict empirical pricing kernels conditional on the 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

epk3VolaIntervalsVDAX

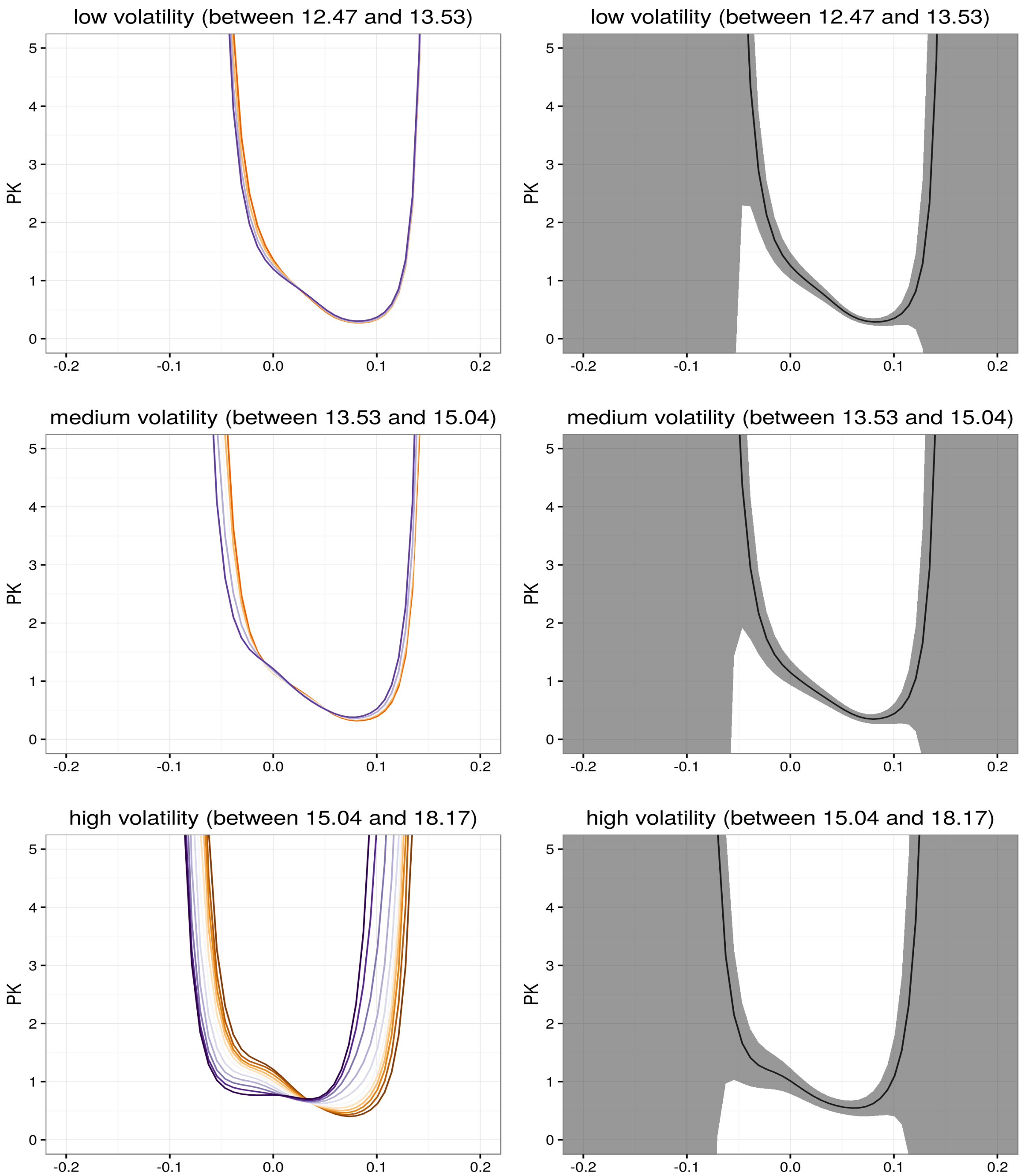

Figure 19: Pricing kernels conditional on time to maturity one month and on different values of VDAX-NEW. Sample period is 1.01.2005 till 31.12.2005. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 7. The panels on the right-hand side depict empirical pricing kernels conditional on the 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

epk3VolaIntervalsVDAX

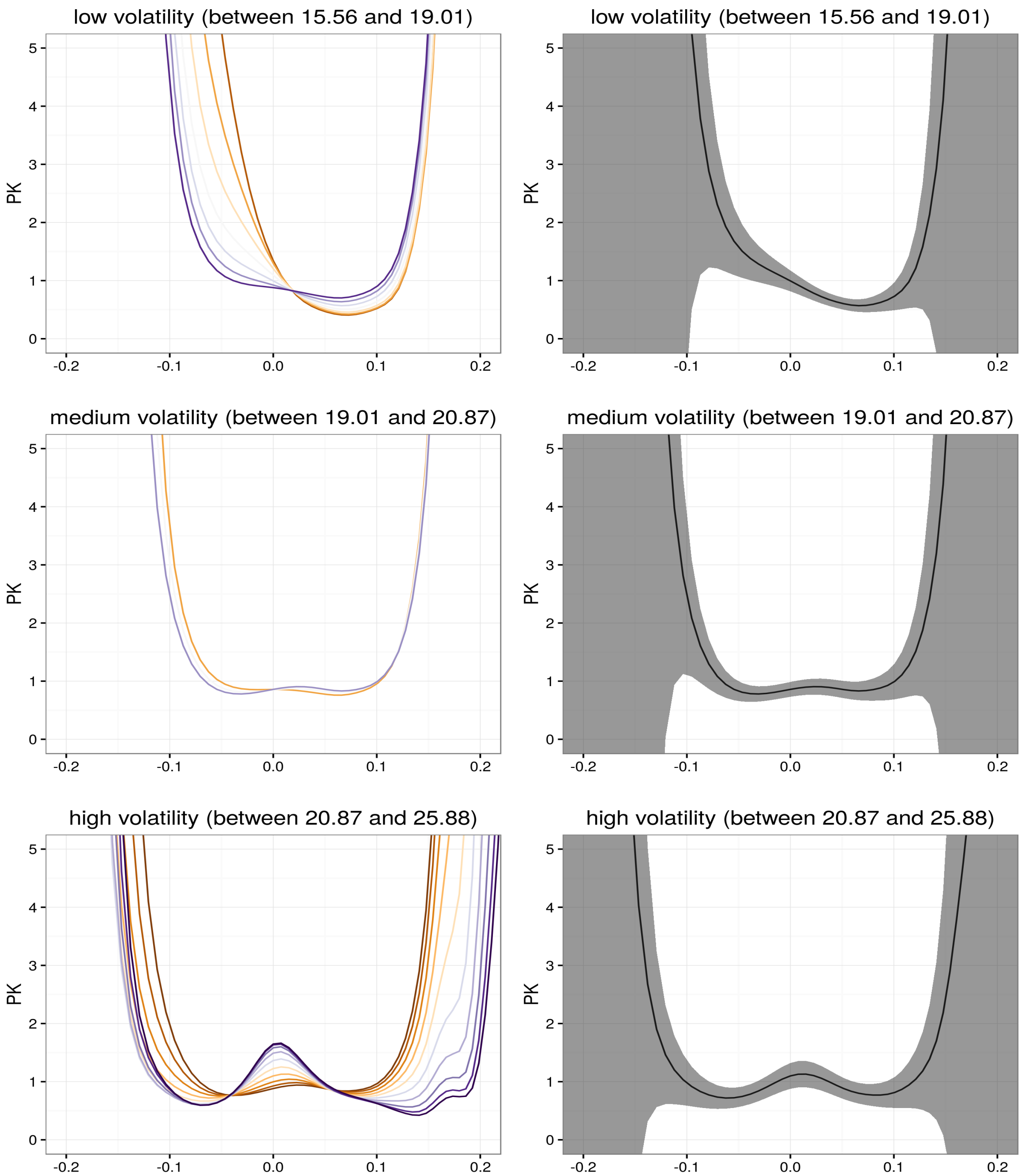

Figure 20: Pricing kernels conditional on time to maturity one month and on different values of VDAX-NEW. Sample period is 1.01.2004 till 31.12.2004. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 7. The panels on the right-hand side depict empirical pricing kernels conditional on the 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

epk3VolaIntervalsVDAX

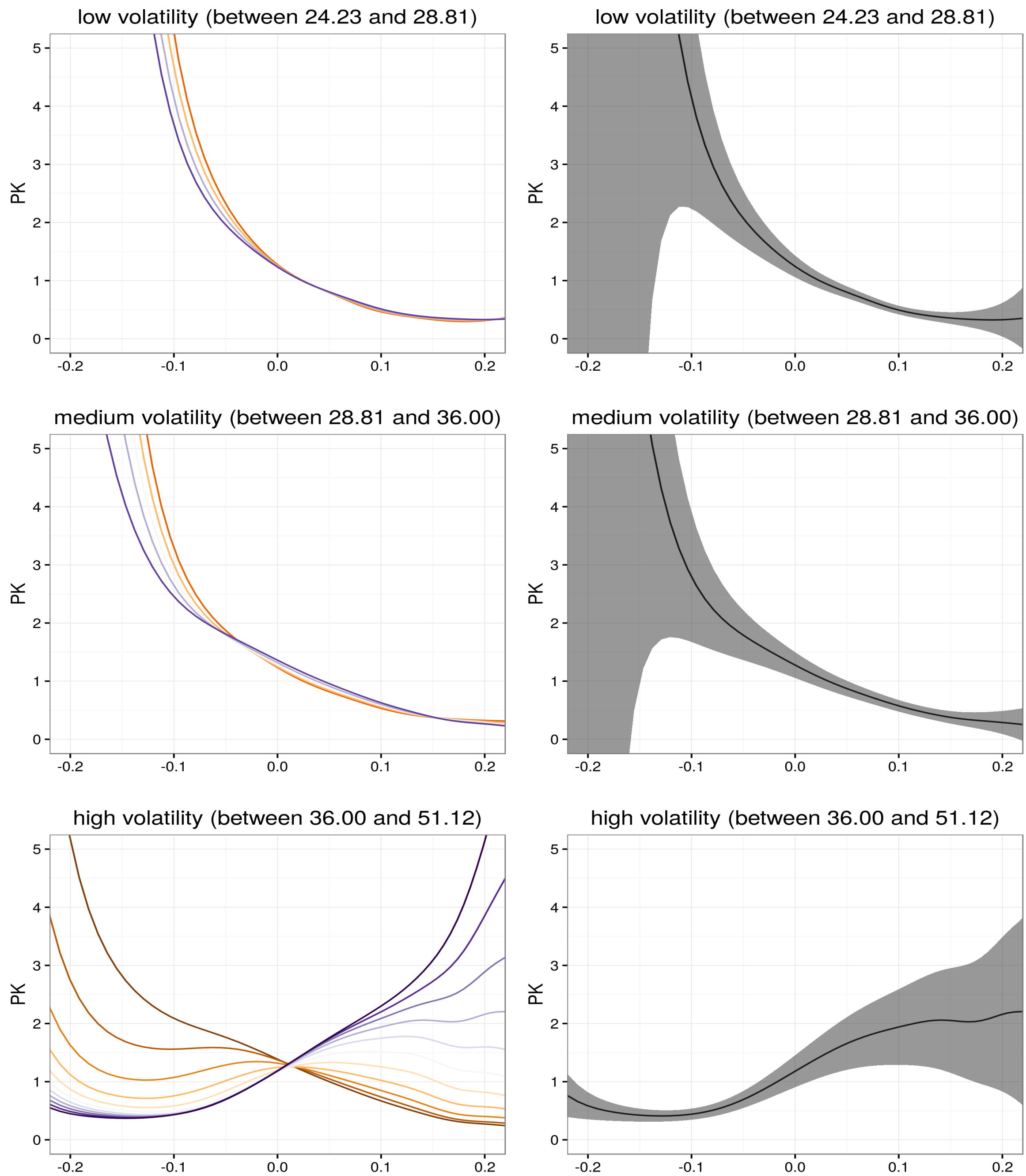

Figure 21: Pricing kernels conditional on time to maturity one month and on different values of VDAX-NEW. Sample period is 1.01.2003 till 31.12.2003. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 7. The panels on the right-hand side depict empirical pricing kernels conditional on the 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

epk3VolaIntervalsVDAX

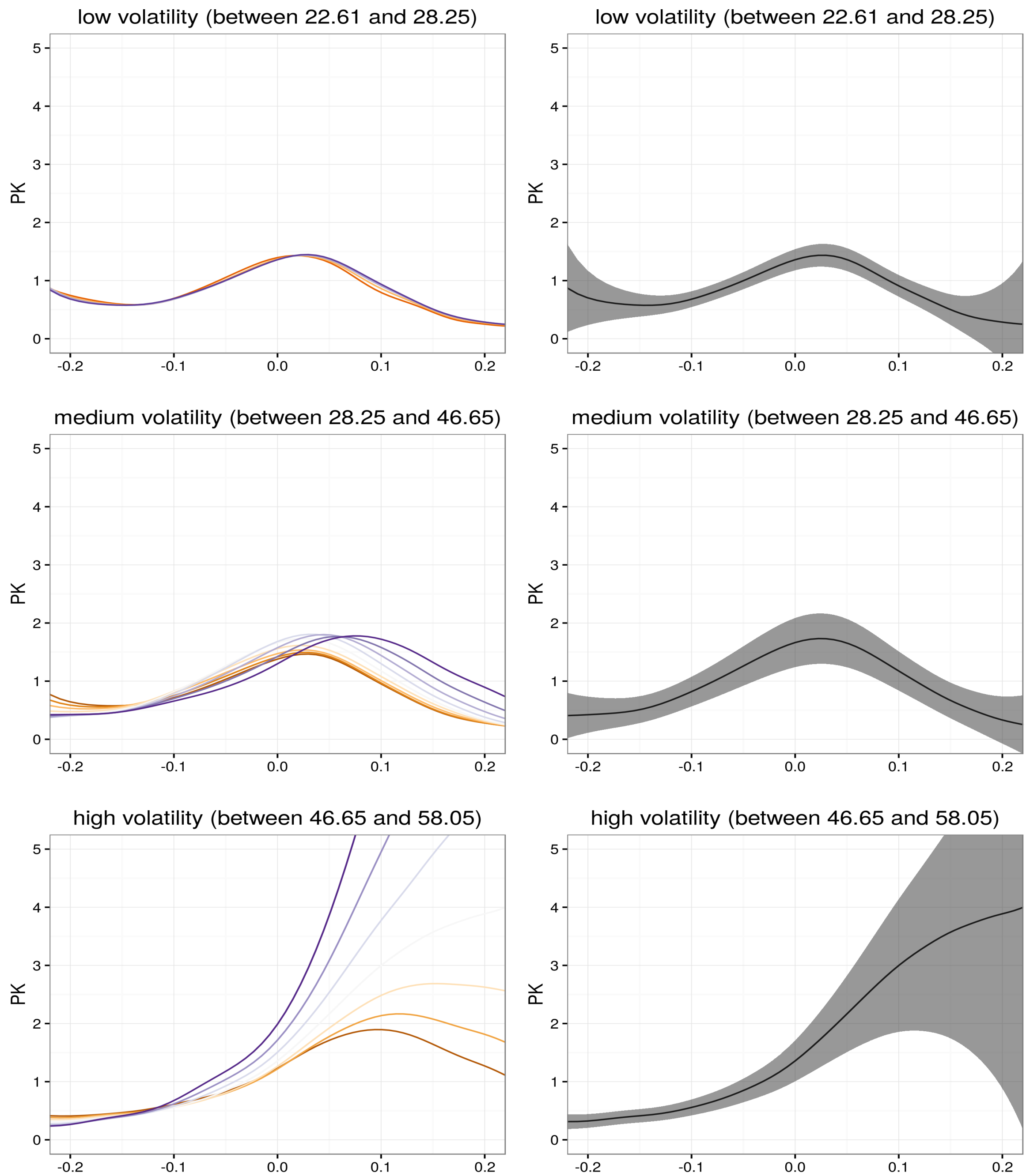

Figure 22: Pricing kernels conditional on time to maturity one month and on different values of VDAX-NEW. Sample period is 1.01.2002 till 31.12.2002. Note: Colors from red to blue correspond to increasing values of volatility within each interval. Endpoints of volatility intervals are presented in Table 7. The panels on the right-hand side depict empirical pricing kernels conditional on the 20%, 50% and 80% quantiles of VDAX-NEW with 95% confidence intervals.

epk3VolaIntervalsVDAX

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